EE 210 ELECTRIC CIRCUITS II

Chapter 10

Sinusoidal Steady-State Power Calculations

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Recap: Sinusoidal Sources

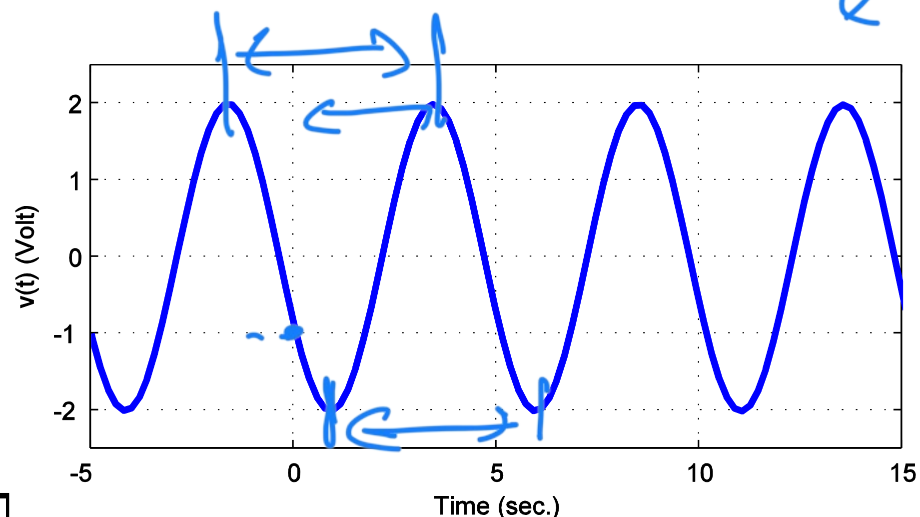
- A **sinusoidal voltage (current) source** produces a voltage (current) that varies sinusoidally in time.

$$v(t) = V_m \cos(\omega t + \phi)$$

Amplitude

Frequency (radians/sec)

Phase



$\omega = 2\pi f$

f : freq $= \frac{1}{T}$

T : period

$$v(t) = V_m \cos(\omega t + \theta)$$

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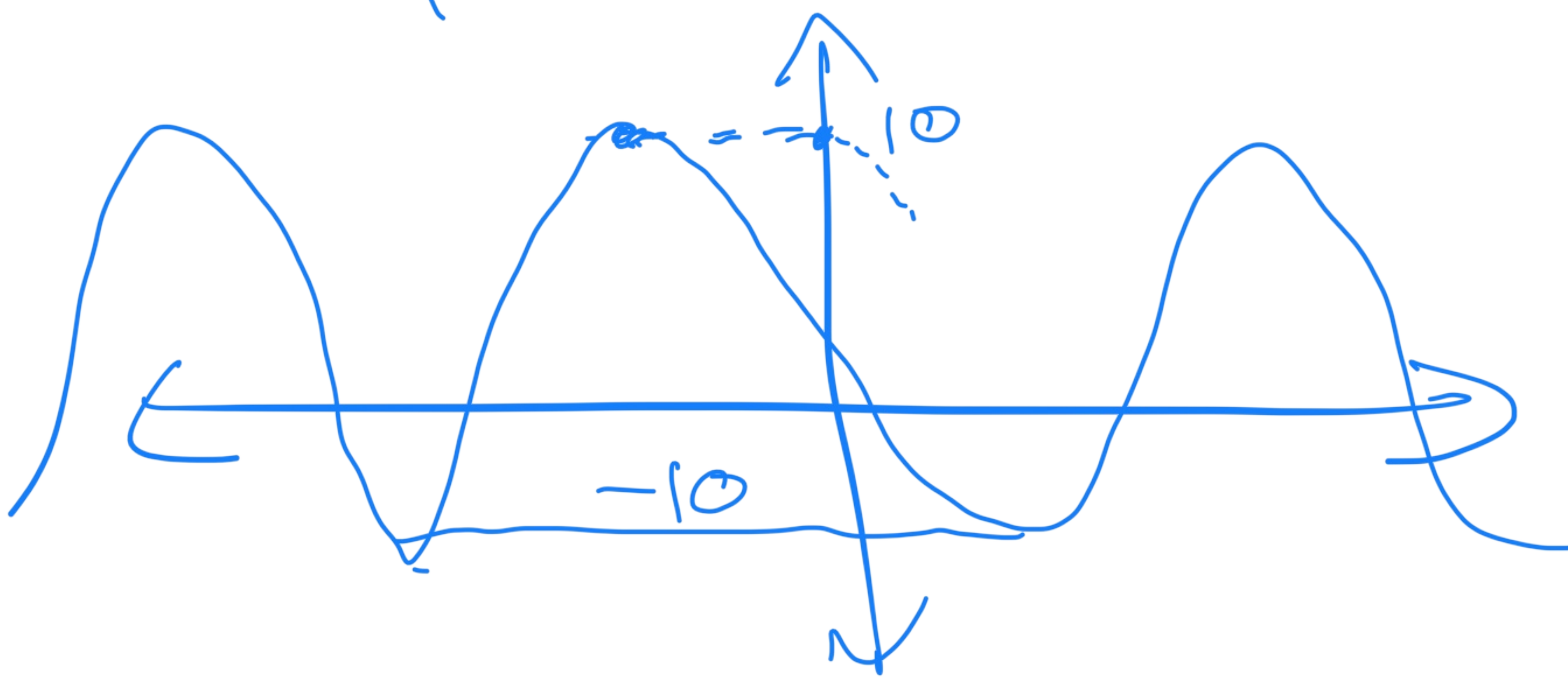


Recap: Frequency and Period

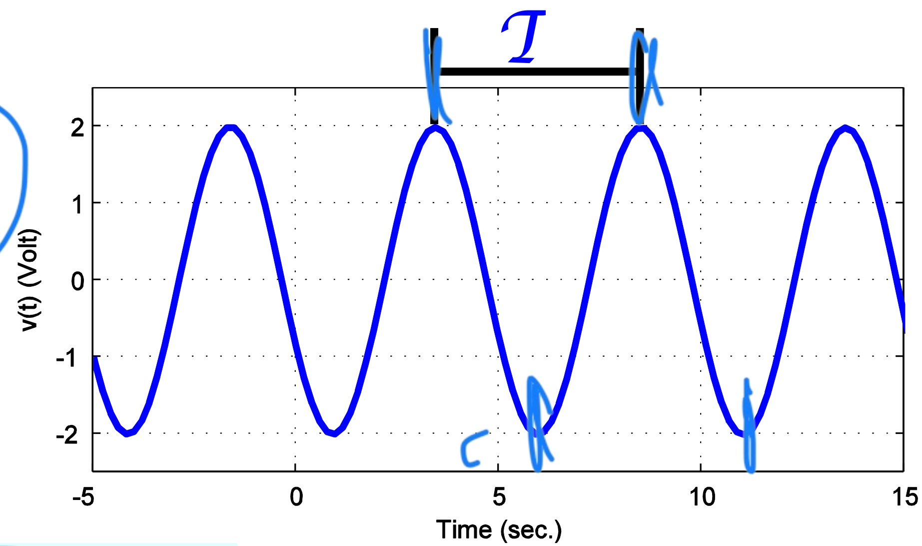
- A function that repeats itself is called **periodic**. The amount of time required to complete one full cycle is called the **period**.
- The number of cycles per second is called the **frequency**.

Period (sec.) → $T = \dfrac{1}{f}$

Hz

Frequency (Hertz) → f

Frequency (rad./sec.) → $\omega = 2\pi f$



Recap: Root-Mean-Square (RMS) Value

- The rms value of a periodic function is defined as the **square-root** of the **mean** value of the **squared** function.

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

- For sinusoidal signal sources, the RMS value can be shown to be

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$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

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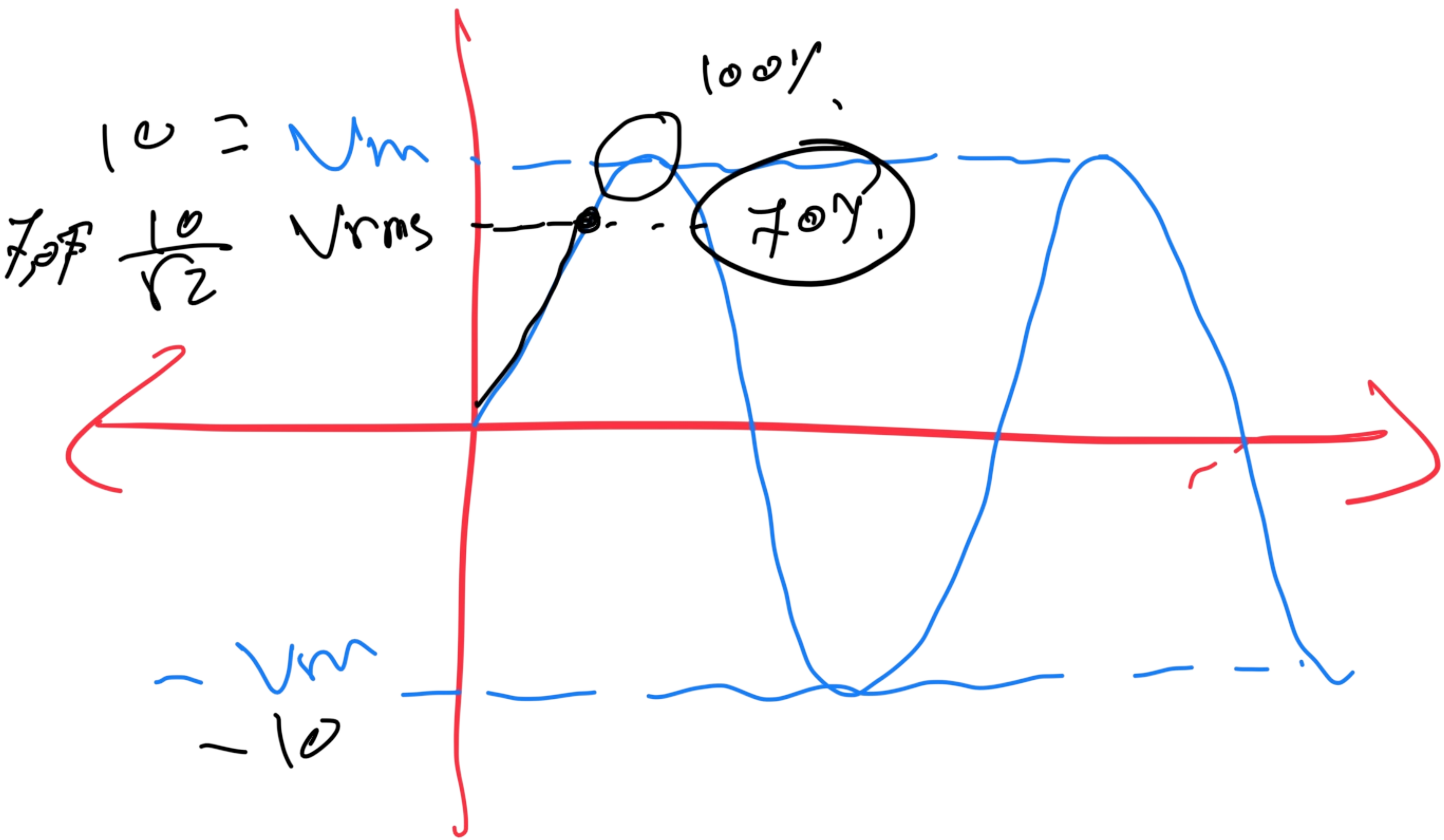
100%
10 = Vm
707 10/√2 Vrms
70%
-Vm
-10

$$\boxed{\begin{gathered} V_{rms} = \frac{V_m}{\sqrt{2}} \\ V_m = \sqrt{2}\, V_{rms} \end{gathered}}$$

$$V_{rms} < \frac{V_m}{}$$

Recap: Phasor

complex power

- **Phasor** is a complex number that carries the **amplitude and phase** information of a sinusoidal function.

time domain

$$\mathcal{V} = \mathcal{P}\left\{V_m \cancel{\cos}(\cancel{\omega t} + \phi)\right\} = \boxed{V_m e^{j\phi}}$$

- The sinusoidal function is obtained from the inverse phasor transform as

$$v(t) = \mathcal{P}^{-1}\left\{\mathcal{V}\right\} = \Re e\left\{\mathcal{V} e^{j\omega t}\right\}$$

Recap: Polar and Rectangular Forms

- **Euler Identity** shows us how to go from polar form to the rectangular form:

$j, i = \sqrt{-1}$

$$e^{j\phi} = cos(\phi) + j\,sin(\phi)$$

- One can then go from rectangular to polar form as

$$V = x + jy = V_m e^{j\phi}$$

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with

$$V_m = \sqrt{x^2 + y^2} \qquad \phi = tan^{-1}\left(\frac{y}{x}\right)$$

Recap: Addition and Multiplication

- Multiplication is more convenient in polar form:

$$\underbrace{\mathcal{V}_{m1}e^{j\phi_1}}_{\mathcal{V}_1} \cdot \underbrace{\mathcal{V}_{m2}e^{j\phi_2}}_{\mathcal{V}_2} = \underbrace{\mathcal{V}_{m1}\mathcal{V}_{m2}}_{\mathcal{V}} e^{j\overbrace{(\phi_1+\phi_2)}^{}\!\!\!\!\!\!_{\phi}}$$

- Taking a power is simple in polar form:

$$\left(\mathcal{V}_m e^{j\phi_1}\right)^k = \mathcal{V}_m^k e^{jk\phi}$$

- Addition is more convenient in rectangular form:

$$\underbrace{\mathcal{V}_{m1}e^{j\phi_1}}_{\mathcal{V}_1} + \underbrace{\mathcal{V}_{m2}e^{j\phi_2}}_{\mathcal{V}_2} = \mathcal{V}_{m1}\cos(\phi_1) + \mathcal{V}_{m2}\cos(\phi_2) + j\left(\mathcal{V}_{m1}\sin(\phi_1) + \mathcal{V}_{m2}\sin(\phi_2)\right)$$

Chapter 10 Sinusoidal Steady-State Power Calculations

In Chapter 9, we calculated the **steady state voltages** and currents in electric circuits driven by **sinusoidal sources.**

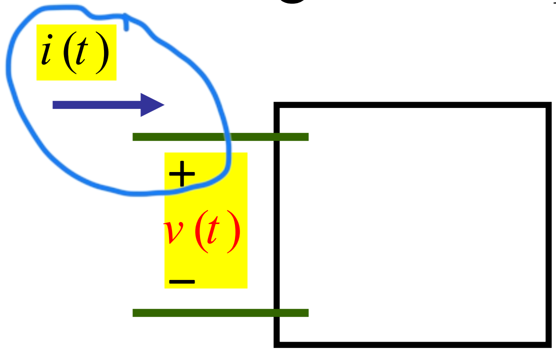
We used **phasor method** to find the steady state **voltages** and currents.

In this Chapter, we consider power in such circuits.

The techniques we develop are useful for analyzing many of the electric devices we encounter daily, because **sinusoidal sources** are predominant means of providing electric power in our homes, school, and businesses.

10.1 Instantaneous Power

Consider the following circuit represented by a black box.



$i(t) = I_m \cos(\omega t + \theta_i)$

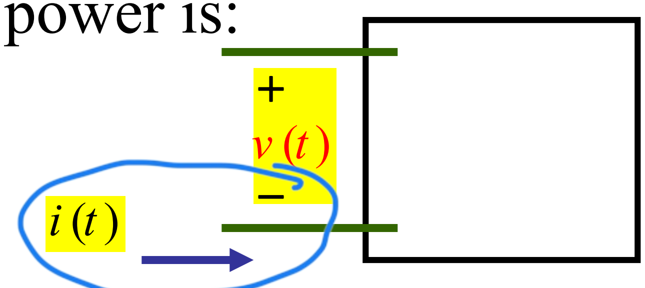
$v(t) = V_m \cos(\omega t + \theta_v)$

The instantaneous power assuming **passive sign convention**

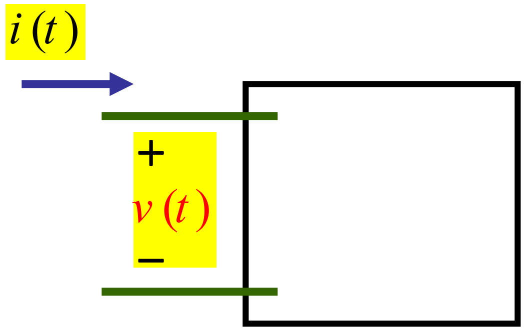
(Current in the direction of **voltage drop** + □ −)

$p(t) = v(t)\, i(t)$ (Watts) +

If the current is in the direction of **voltage rise (− □ +)** the instantaneous power is:



$p(t) = -v(t)\, i(t)$



$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t)$$

$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i)$$

$$p(t) = v(t)\,i(t) = \{V_m \cos(\omega t + \theta_v - \theta_i)\}\{I_m \cos(\omega t)\}$$
$$= V_m I_m \cos(\omega t + \theta_v - \theta_i)\cos(\omega t)$$

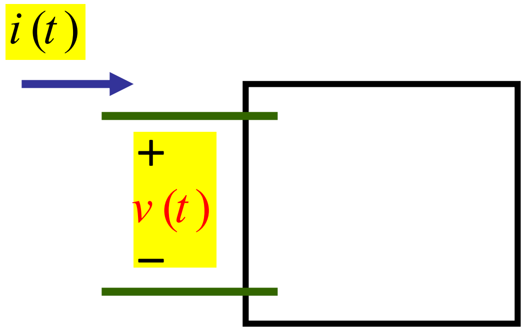
Since

$$\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$$

Therefore

$$p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$



$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$\Rightarrow$$

$$i(t) = I_m \cos(\omega t)$$

$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i)$$

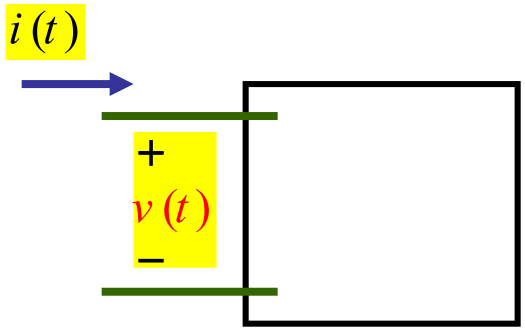
Therefore $$p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

Since $$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Therefore $$\cos(2\omega t + \theta_v - \theta_i) = \cos(\theta_v - \theta_i)\cos(2\omega t) - \sin(\theta_v - \theta_i)\sin(2\omega t)$$

$$\Rightarrow$$

$$\boxed{p(t) = \underbrace{\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)}_{\text{Average power}} + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin(2\omega t)}$$

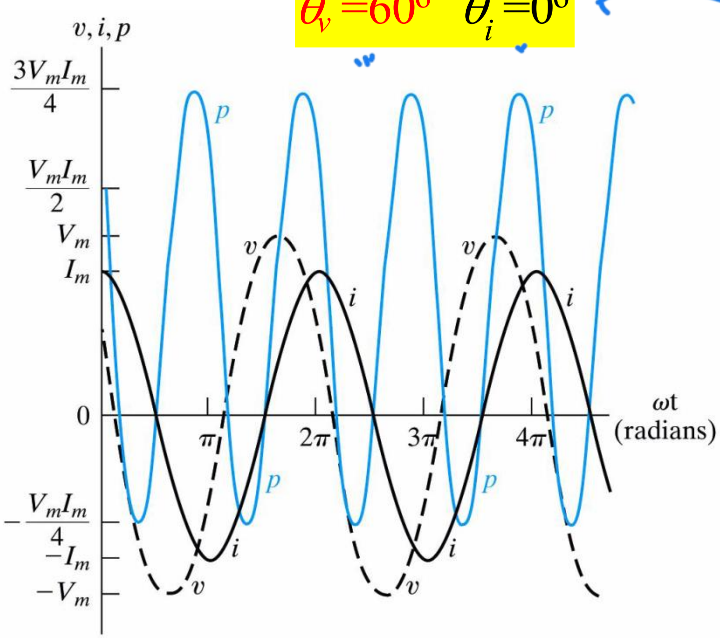


$$i(t) = I_m \cos(\omega t)$$

$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i)$$

$$p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin(2\omega t)$$

$\theta_v = 60^o \quad \theta_i = 0^o$



You can see that the frequency of the Instantaneous power is twice the frequency of the voltage or current

10.2 Average and Reactive Power

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Recall the Instantaneous power $p(t)$

$$p(t)=\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)+\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)\cos(2\omega t)-\frac{V_m I_m}{2}\sin(\theta_v-\theta_i)\sin(2\omega t)$$

$$p(t)=P+P\cos(2\omega t)-Q\sin(2\omega t)$$

بسيط

where

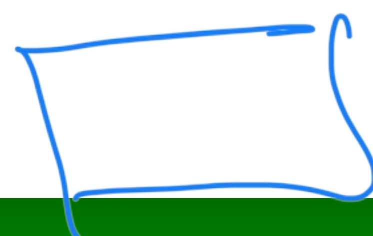
$$P=\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)$$

Average Power (**Real Power**) فعالة

$$Q=\frac{V_m I_m}{2}\sin(\theta_v-\theta_i)$$

Reactive Power قدرة غير فعالة

الرغوة



10.2 Average and Reactive Power

Average Power P is sometimes called **Real power** because it describes the power in a circuit that is transformed from **electric** to **non electric** (**Example Heat**).

It is easy to see why P is called Average Power because

$$\frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} \{P + P\cos(2\omega t) - Q\sin(2\omega t)\}dt \quad = P$$

Power for purely resistive Circuits

R $\Rightarrow$ $\boxed{\theta_v = \theta_i}$

$$p(t) = P + P\cos(2\omega t) - Q\sin(2\omega t)$$

$\theta_v = \theta_i$ ➡ $P = \dfrac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \dfrac{V_m I_m}{2}\cos(0) = \dfrac{V_m I_m}{2}$

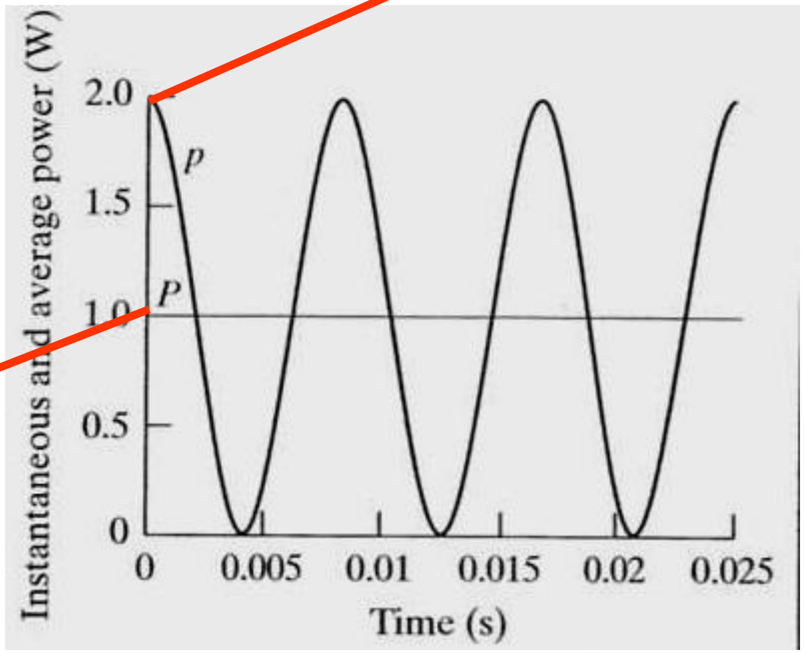
$Q = \dfrac{V_m I_m}{2}\sin(\theta_v - \theta_i) = \dfrac{V_m I_m}{2}\sin(0)$ $= 0$

➡ $p(t) = \dfrac{V_m I_m}{2} + \dfrac{V_m I_m}{2}\cos(2\omega t)$

The instantaneous power can never be negative.

$V_m I_m$

$\dfrac{V_m I_m}{2}$



➡ Power can not be extracted from a purely resistive network.

Power for purely Inductive Circuits

L → P = 0, Q =

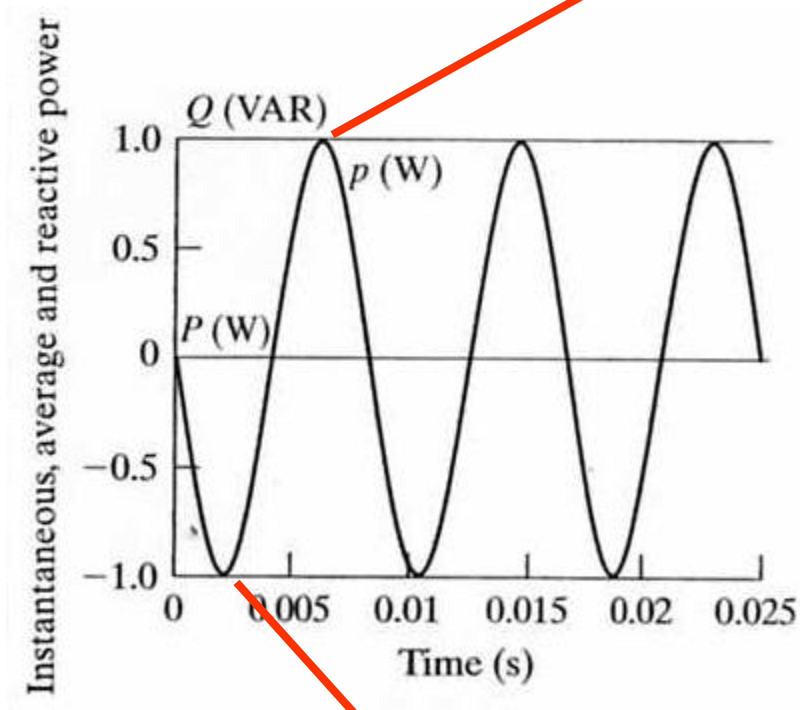
$$p(t) = P + P\cos(2\omega t) - Q\sin(2\omega t) \qquad \theta_v = \theta_i + 90^o$$

$$\Rightarrow \theta_v - \theta_i = 90^o \Rightarrow P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$

$$= \frac{V_m I_m}{2}\cos(90^o) = 0$$

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i) = \frac{V_m I_m}{2}\sin(90^o) = \frac{V_m I_m}{2}$$

$$\Rightarrow p(t) = -\frac{V_m I_m}{2}\sin(2\omega t)$$



$\frac{V_m I_m}{2}$

$-\frac{V_m I_m}{2}$

The instantaneous power p(t) is continuously **exchanged** between the circuit and the source driving the circuit. **The average power is zero.**

P = 0

Power for purely Inductive Circuits

P

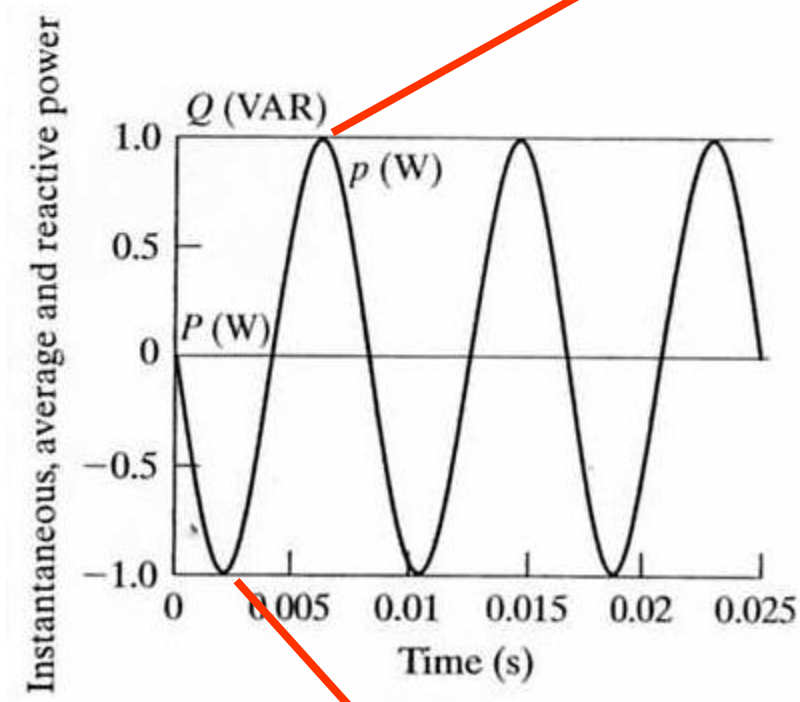
P + → stored

P − → extracted

When $p(t)$ is **positive**, energy is being **stored** in the **magnetic field** associated with the **inductive** element.

$\frac{V_m I_m}{2}$

When $p(t)$ is **negative**, energy is being **extracted** from the **magnetic field.**



$-\frac{V_m I_m}{2}$

The power associated with **purely inductive** circuits is the reactive power Q.

The dimension of **reactive power** Q is the same as the average power P. To distinguish them we use the unit **VAR** (**V**olt **A**mpere **R**eactive) for **reactive power.**

P = W

Q = VAR

Power for purely Capacitive Circuits

C → P = 0

Q = ...

$$p(t) = P + P\cos(2\omega t) - Q\sin(2\omega t) \qquad \theta_v = \theta_i - 90^o$$

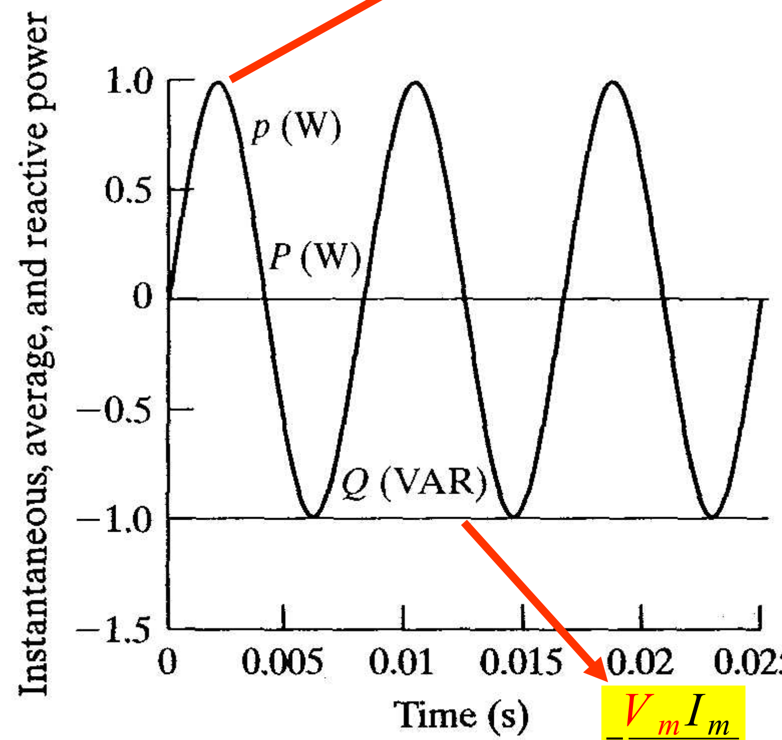
➡ $\theta_v - \theta_i = -90^o$ ➡ $P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{V_m I_m}{2}\underbrace{\cos(-90^o)}_{0} = 0$ $\qquad \frac{V_m I_m}{2}$

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

$$= \frac{V_m I_m}{2}\sin(-90^o) = -\frac{V_m I_m}{2}$$

➡ $$p(t) = +\frac{V_m I_m}{2}\sin(2\omega t)$$

The instantaneous power p(t) is continuously **exchanged** between the circuit and the source driving the circuit. **The average power is zero.**



$-\frac{V_m I_m}{2}$

Power for purely Capacitive Circuits

$P+ \rightarrow$ stored

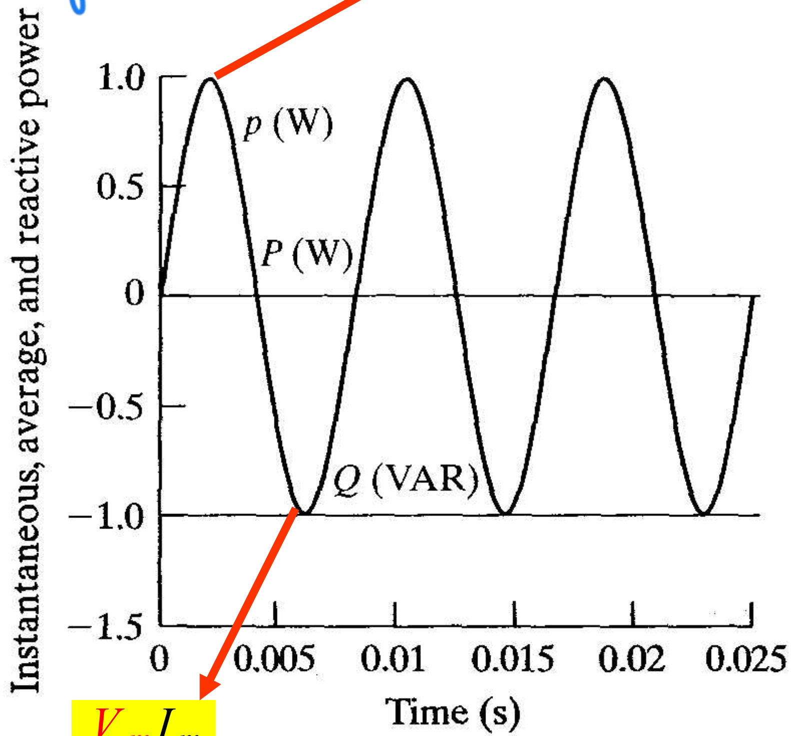
$P- \rightarrow$ extracted

When p(t) is **positive**, energy is being **stored** in the **electric field** associated with the **capacitive** element.

When p(t) is **negative**, energy is being **extracted** from the **electric field.**

The power associated with **purely capacitive** circuits is the reactive power Q (**VAR**).

$\frac{V_m I_m}{2}$



$-\frac{V_m I_m}{2}$

The power factor

Recall the Instantaneous power $p(t)$

$$p(t)=\underbrace{\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)}_{P\ \textbf{average power}}+\underbrace{\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)}_{P\ \textbf{average power}}\cos(2\omega t)-\underbrace{\frac{V_m I_m}{2}\sin(\theta_v-\theta_i)}_{Q\ \textbf{reactive power}}\sin(2\omega t)$$

$$= P + P\cos(2\omega t) - Q\sin(2\omega t)$$

The angle $\theta_v-\theta_i$ plays a role in the computation of both **average** and **reactive** power

The angle $\theta_v-\theta_i$ is referred to as the **power factor angle**

We now define the following : The **power factor** $\textbf{pf}=\cos(\theta_v-\theta_i)$

The **reactive factor** $\textbf{rf}=\sin(\theta_v-\theta_i)$

The **power factor**

$$\mathbf{pf} = \cos(\theta_v - \theta_i)$$

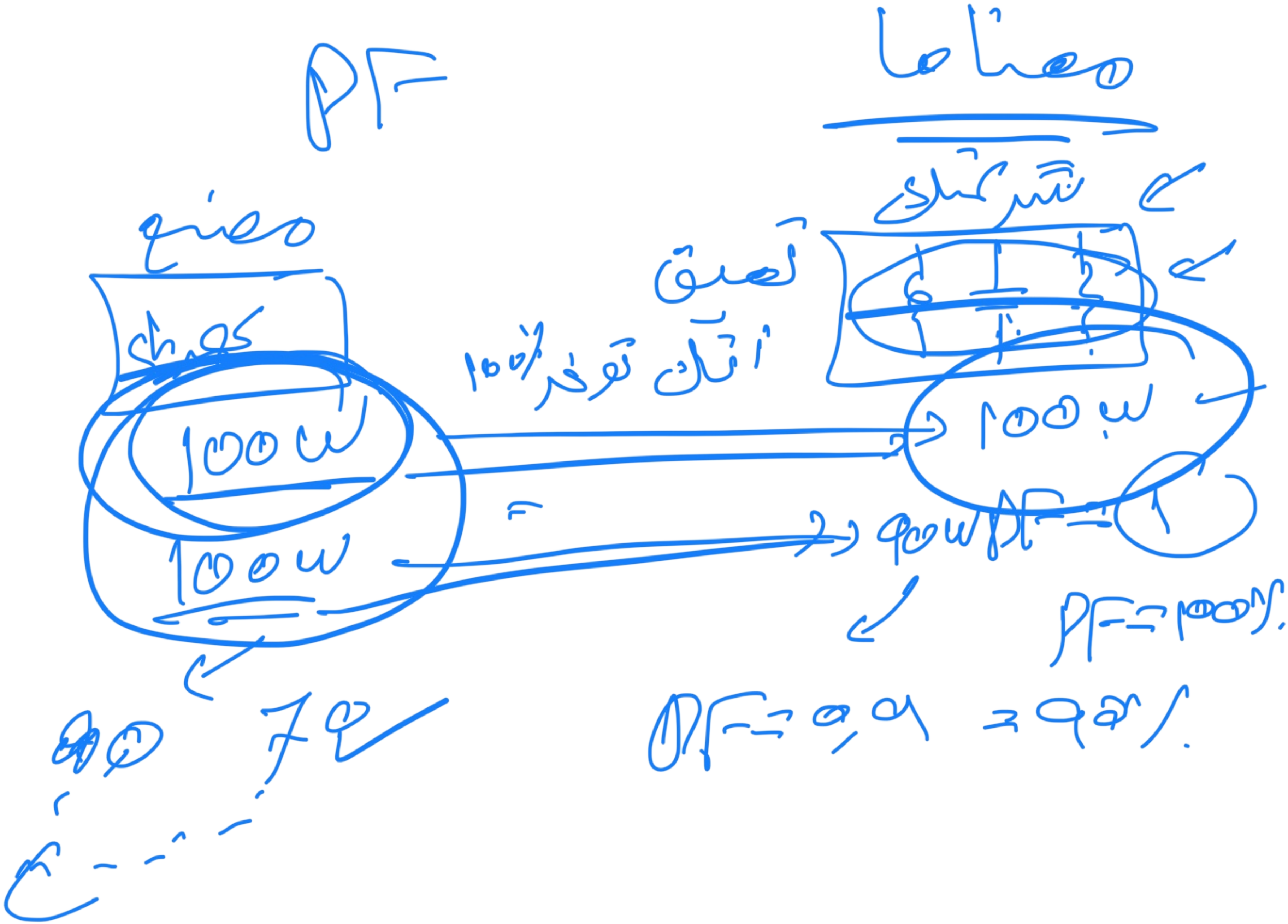
Knowing the power factor **pf** does not tell you the power factor angle, because

$$\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$$

To completely describe this angle, we use the descriptive phrases **lagging power factor** and **leading power factor**

Lagging power factor implies that current **lags** voltage hence an inductive load

Leading power factor implies that current **leads** voltage hence a capacitive load



PF
مصنع
10%
100w
100w
100w
90w
PF = 1
PF = 100%
PF = 0.9 = 90%

Average power [illegible]

$$P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

$$0 \leq PF \leq 1$$

$$PF = \cos(\theta_v - \theta_i)$$

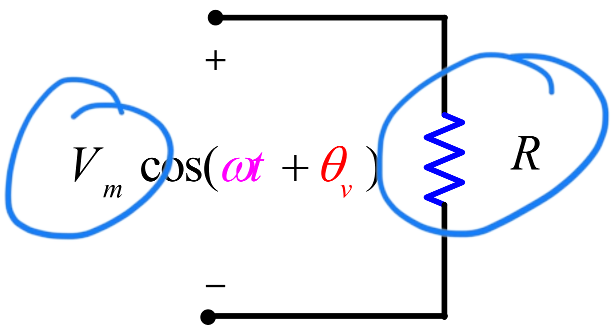
$\theta_v = \theta_i \rightarrow$ purely resistive $R \rightarrow PF = 1$

$\theta_v > \theta_i \rightarrow$ inductor $\Rightarrow PF$ lagging

$\theta_v < \theta_i \rightarrow$ capacitive $\Rightarrow PF$ leading

10.3 The rms Value and Power Calculations

Assume that a sinusoidal voltage is applied to the terminals of a resistor as shown



$P = vi = \frac{v^2}{R} = i^2 R$

Suppose we want to determine the average power delivered to the resistor

10.3 The rms Value and Power Calculations

Suppose we want to determine the average power delivered to the resistor

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{\left\{V_m \cos(\omega t + \theta_v)\right\}^2}{R} dt = \frac{1}{R}\left[\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \theta_v) dt\right]$$

However since $V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \theta_v) dt}$

$$P = \frac{V_{rms}^2}{R}$$

If the resistor carry sinusoidal current $P = RI_{rms}^2$

Recall the Average and Reactive power

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) \qquad Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

Which can be written as

$$P = \frac{V_m I_m}{\sqrt{2}\sqrt{2}}\cos(\theta_v - \theta_i) \qquad Q = \frac{V_m I_m}{\sqrt{2}\sqrt{2}}\sin(\theta_v - \theta_i)$$

Therefore the Average and Reactive power can be written in terms of the **rms** value as

$$P = V_{\text{rms}} I_{\text{rms}}\cos(\theta_v - \theta_i) \qquad Q = V_{rms} I_{rms}\sin(\theta_v - \theta_i)$$

The **rms** value is also referred to as the **effective value** **eff**

Therefore, the average and reactive power can be written in terms of the **eff** value as:

$$P = V_{\text{eff}} I_{\text{eff}}\cos(\theta_v - \theta_i) \qquad Q = V_{\text{eff}} I_{\text{eff}}\sin(\theta_v - \theta_i)$$

eff
rms

$$V_{rms} = \frac{V_m}{\sqrt{2}} \longrightarrow V_m = \sqrt{2}\,V_{rms}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}} \longrightarrow I_m = \sqrt{2}\,I_{rms}$$

Average power $P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$

$$\boxed{\begin{aligned} P &= \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \\ P &= V_{rms} I_{rms} \cos(\theta_v - \theta_i) \end{aligned}}$$

Example 10.3

a) A sinusoidal voltage having a maximum amplitude of 625 V is applied to the terminals of 50 Ω resistor. Find the average power delivered to the resistor.

b) Repeat (a) by first finding the current in the resistor.

V_m

a) $V_m = 625\ V$ ✓ $R = 50\ \Omega$

$V_{rms} = \frac{625}{\sqrt{2}} = 441{,}9\ V$

$P = \frac{V_{rms}^2}{R}$

$P = \frac{441{,}9^2}{50} = 3906{,}25\ W$

b) $V = IR \rightarrow I_m = \frac{V_m}{R}$

$$I_m = \frac{625}{50} = 12{,}5\ A$$

$$P = \frac{V_m I_m}{2} \underbrace{\cos(\theta_v - \theta_i)}_{1}$$

$$P = \frac{625 \times 12{,}5}{2} = \boxed{3906{,}25\,W}$$

Example 10.3

Answer:

a) P=3906.25 W. b) I=12.5 A, P=3906.25 W

10.4 Complex Power

Previously, we found it convenient to introduce sinusoidal voltage and current in terms of the complex number, the **phasor.**

Definition

Let the **complex power** be the complex sum of real power and reactive power

$$S = P + jQ$$

were

S is the complex power

P is the average power

Q is the reactive power

$j = i = \sqrt{-1}$

$S = P + jQ$

Imag number

عدد تخيلي

Real number

عدد حقيقي

$P = Re\{S\}$, $Q = Im\{S\}$

$S = 500 + j\,1400$

$P = 500\ W$ $\quad$ $Q = 1400\ VAR$

Advantages of using complex power

$$\boldsymbol{S} = P + jQ$$

– We can compute the average and reactive power from the complex power $\boldsymbol{S}$

$$P = \Re\{\boldsymbol{S}\} \qquad Q = \Im\{\boldsymbol{S}\}$$

– complex power $\boldsymbol{S}$ provide a geometric interpretation

$$\boldsymbol{S} = P + jQ = |\boldsymbol{S}|\, e^{j\theta}$$

where

$|\boldsymbol{S}| = \sqrt{P^2 + Q^2}$ is called the apparent power

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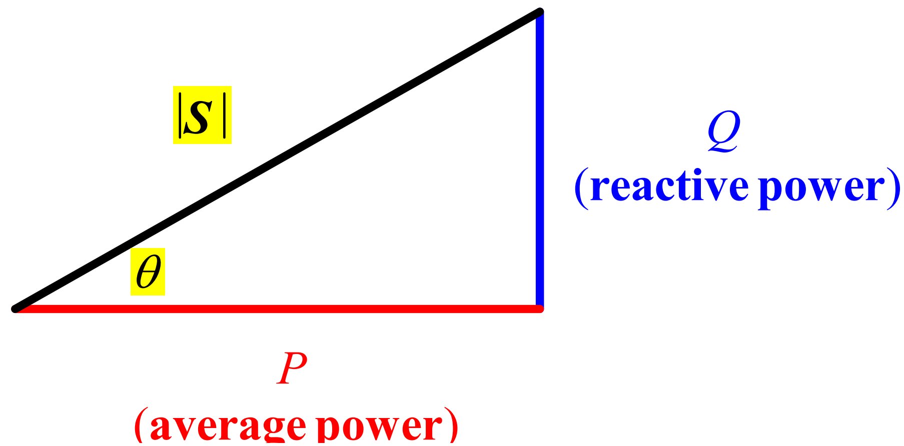
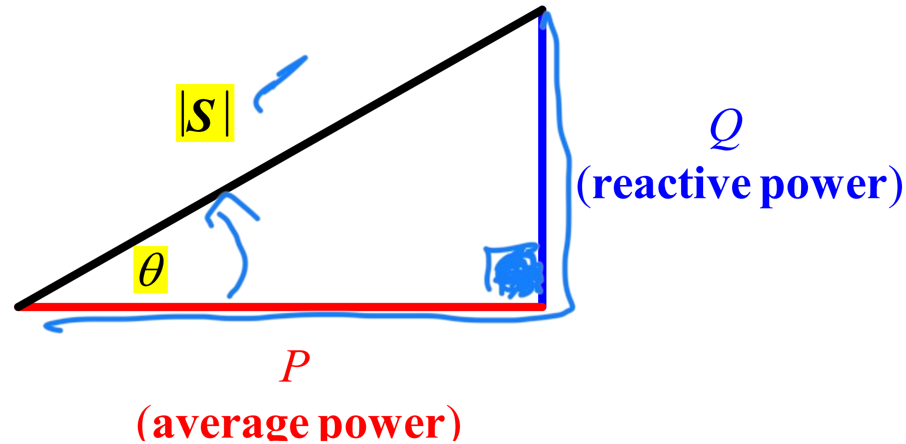


Advantages of using complex power

$\boldsymbol{S} = P + jQ$

$S = P + jQ$

$= |S| \angle \theta$



$|\boldsymbol{S}| = \sqrt{P^2 + Q^2}$ is called the apparent power

$$\theta = \tan^{-1}\left(\frac{Q}{P}\right) = \tan^{-1}\left(\frac{V_m I_m \sin(\theta_v - \theta_i)}{V_m I_m \cos(\theta_v - \theta_i)}\right) = \tan^{-1}\left(\frac{\sin(\theta_v - \theta_i)}{\cos(\theta_v - \theta_i)}\right) = \tan^{-1}\left(\tan(\theta_v - \theta_i)\right) = \underbrace{\theta_v - \theta_i}_{\text{power factor angle}}$$

The geometric relations for a right triangle mean the four power triangle dimensions
($|\boldsymbol{S}|$, P, Q, θ) can be determined if **any two** of the four are known.

Example 10.4

An electrical load operates at 240 V rms. The load absorbs an average power of 8 kW at a lagging power factor of 0.8.

a) Calculate the complex power of the load.

b) Calculate the impedance of the load.

a) P=8kW, → S =P/cos 36.87=10 kVA

$$\mathbf{Q}=8\tan(36.87)=6kVAR \qquad \mathbf{S}=P+jQ=8+j6kVA$$

b)

$$P=V_{eff}I_{eff}\cos(\theta_v-\theta_i)=(240)I_{eff}(0.8)=8000W$$

$$\Rightarrow I_{eff}=41.67A$$

$$Z=\frac{V_{eff}}{I_{eff}}=\frac{240}{41.67}=5.76\Omega$$

$V_{rms} = 240\ V$

$\boxed{P = 8\ kW}$

$PF = 0{,}8$ lagging

a) $S = P + jQ$

$PF = \cos\theta$

$P = V_{rms} I_{rms} PF \Rightarrow I_{rms} = \dfrac{P}{V_{rms} PF}$

$I_{rms} = \dfrac{8000}{240 \times 0{,}8} = \dfrac{125}{3}\ A \Rightarrow 41{,}67A$

$$Q = V_{rms} I_{rms} \sin(\theta)$$

$$\theta = \cos^{-1}(PF)$$

$$PF = \cos\theta \rightarrow \theta = \cos^{-1}(PF)$$

$$Q = 240 \times \frac{125}{3} \sin(\cos^{-1}(0.8))$$

$$\boxed{Q = 6000 \text{ VAR}}$$

$$\boxed{S = 8 + j6 \text{ KVA}}$$

b) $Z = \dfrac{V_{rms}}{I_{rms}} \rightarrow \dfrac{240}{125/3}$

$$\boxed{Z = 5.76\ \Omega}$$

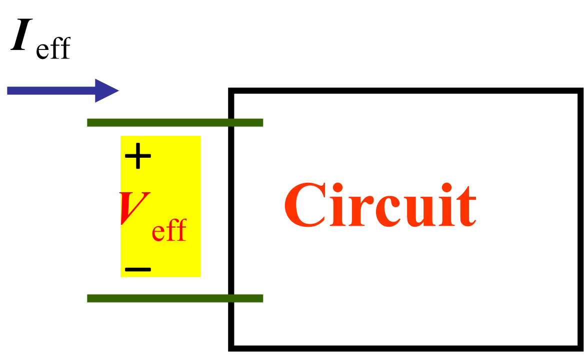
10.5 Power Calculations

$$\boldsymbol{S} = P + jQ = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + j\,\frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

$$= \frac{V_m I_m}{2}\left[\cos(\theta_v - \theta_i) + j\,\sin(\theta_v - \theta_i)\right] = \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} = V_{\text{eff}} I_{\text{eff}} e^{j(\theta_v - \theta_i)}$$

$$= V_{\text{eff}} e^{j\theta_v} I_{\text{eff}} e^{-j\,\theta_i} \qquad \boxed{= \boldsymbol{V}_{\text{eff}} \boldsymbol{I}^*_{\text{eff}}}$$

where $\boldsymbol{I}^*_{\text{eff}}$ Is the conjugate of the current phasor $\boldsymbol{I}_{\text{eff}}$



Also $\quad \boldsymbol{S} = \frac{1}{2}\,\boldsymbol{V}\boldsymbol{I}^*$

$$S = P + jQ \rightarrow \text{T}$$

$$P = V_{rms} I_{rms} \underline{\underline{\cos(\theta_v - \theta_i)}} \rightarrow PF$$

$$Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

$$\boxed{S = \frac{1}{2} V_m I_m^* = V_{rms} I_{rms}^*}$$

Complex number

$$Z = 2 + 2i$$

$$Z = 2\sqrt{2} \angle 45$$

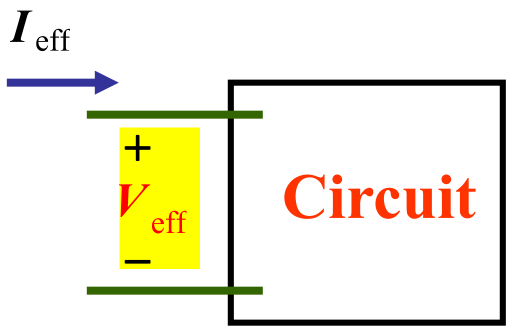
$\rightarrow Z^* = 2 - 2i$

$\rightarrow Z^* = 2\sqrt{2} \angle -45$

$* \rightarrow$ conjugate $\rightarrow$ تغير اشارة i

أو اشارة الزاوية .

Alternate Forms for Complex Power



The complex power was **defined** as

$$S = P + jQ$$

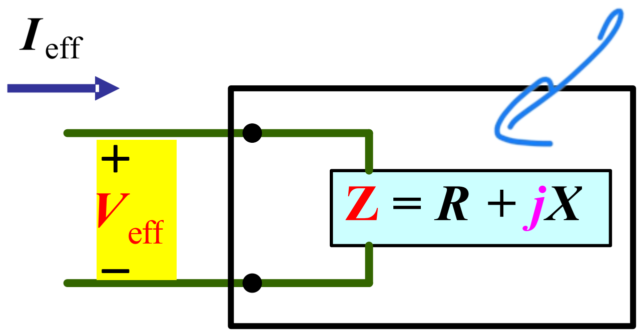
Then complex power was **calculated** to be

$$\boxed{S = V_{\text{eff}} I^*_{\text{eff}}} \quad \textbf{OR} \quad \boxed{S = \frac{1}{2} V I^*}$$

eff = rms

However there are several useful variations as follows:

First variation



$$S = V_{\text{eff}} I^*_{\text{eff}} = (Z I_{\text{eff}}) I^*_{\text{eff}} = Z I_{\text{eff}} I^*_{\text{eff}} \boxed{= Z |I_{\text{eff}}|^2}$$

$$= (R + jX)|I_{\text{eff}}|^2 = \underbrace{R\,|I_{\text{eff}}|^2}_{P} + \underbrace{jX\,|I_{\text{eff}}|^2}_{Q}$$

$$P = R\,|I_{\text{eff}}|^2 = R I^2_{\text{eff}} = \frac{1}{2} R I^2_{m}$$

$$Q = X\,|I_{\text{eff}}|^2 = X I^2_{\text{eff}} = \frac{1}{2} X I^2_{m}$$

$$S = P + jQ \qquad Z = R + jX$$

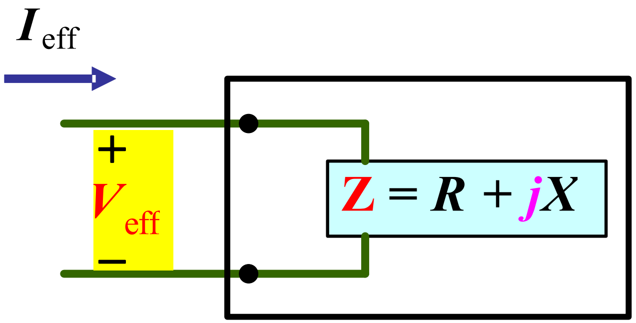
$+jX \rightarrow$ inductor

$-jX \rightarrow$ capacitor

$$P = |I_{eff}|^2 R = \frac{|V_{eff}|^2}{R}$$

$$Q = |I_{eff}|^2 X = \frac{|V_{eff}|^2}{X}$$

Second variation



$$S = V_{\text{eff}} I^*_{\text{eff}} = V_{\text{eff}} \left(\frac{V_{\text{eff}}}{Z} \right)^* = \frac{V_{\text{eff}} V^*_{\text{eff}}}{Z^*} = \boxed{\frac{|V_{\text{eff}}|^2}{Z^*}}$$

$$= \frac{|V_{\text{eff}}|^2}{R - jX} = \frac{|V_{\text{eff}}|^2}{R - jX} \frac{R + jX}{R + jX} = \frac{R + jX}{R^2 + X^2} |V_{\text{eff}}|^2$$

$$= \underbrace{\frac{R}{R^2 + X^2} |V_{\text{eff}}|^2}_{P} + j \underbrace{\frac{X}{R^2 + X^2} |V_{\text{eff}}|^2}_{Q}$$

$$\Rightarrow \quad P = \frac{R}{R^2 + X^2} |V_{\text{eff}}|^2 = \frac{R}{R^2 + X^2} V^2_{\text{eff}} = \frac{1}{2} \frac{R}{R^2 + X^2} V^2_{\text{m}}$$

$$Q = \frac{X}{R^2 + X^2} |V_{\text{eff}}|^2 = \frac{X}{R^2 + X^2} V^2_{\text{eff}} = \frac{1}{2} \frac{X}{R^2 + X^2} V^2_{\text{m}}$$

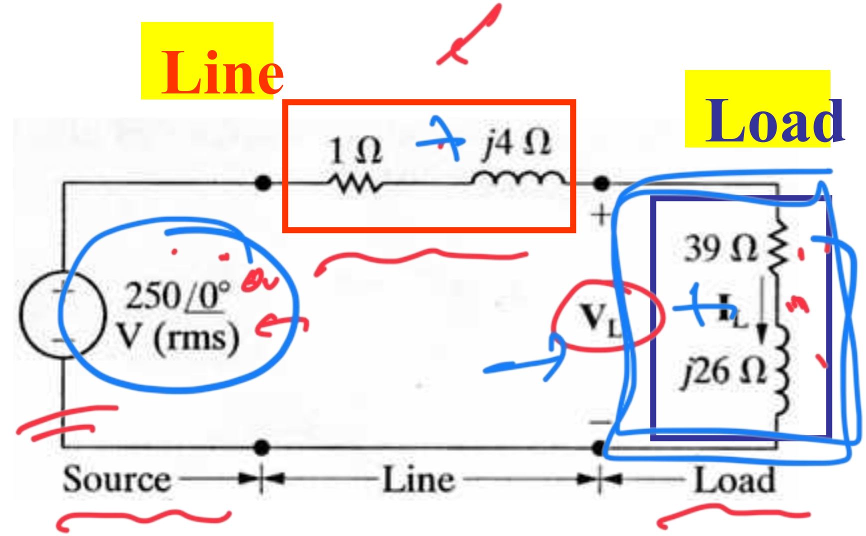
If $Z = R$ (pure resistive) $X = 0$ $\Rightarrow$ $P = \frac{R}{R^2 + X^2} |V_{\text{eff}}|^2 = \frac{|V_{\text{eff}}|^2}{R}$ $\quad Q = 0$

If $Z = X$ (pure reactive) $R = 0$ $\Rightarrow$ $P = 0$ $\quad Q = \frac{X}{R^2 + X^2} |V_{\text{eff}}|^2 = \frac{|V_{\text{eff}}|^2}{X}$

Example 10.5

In the circuit shown a load having an impedance of $39 + j26\ \Omega$ is fed from a voltage source through a line having an impedance of $1 + j4\ \Omega$. The effective, or rms, value of the source voltage is 250 V.

a) Calculate the load current $\mathbf{I}_L$ and voltage $\mathbf{V}_L$.



SOLUTION

a) The line and load impedances are in series across the voltage source, so the load current equals the voltage divided by the total impedance, or

$$\mathbf{I}_L = \frac{250\angle 0^\circ}{40 + j30} = 4 - j3 = 5\angle -36.87^\circ \text{ A (rms)}$$

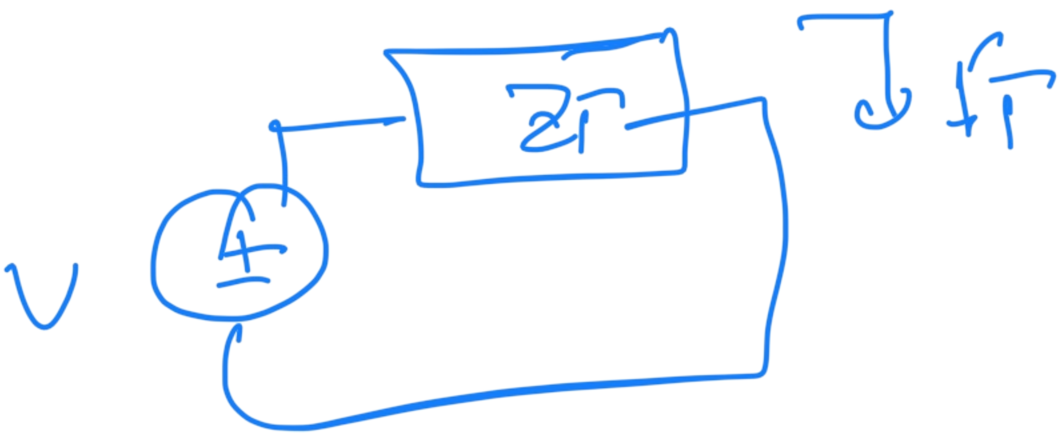
rms because the voltage is given in terms of **rms.**

$$\mathbf{V}_L = (39 + j26)\mathbf{I}_L = 234 - j13 = 234.36\angle -3.18^\circ \text{ V (rms)}$$

Series $\rightarrow$ $Z = Z_L + Z_{load}$

$$Z = 1 + j4 + 39 + j26$$

$$\boxed{Z_T = 40 + j30\ \Omega}$$



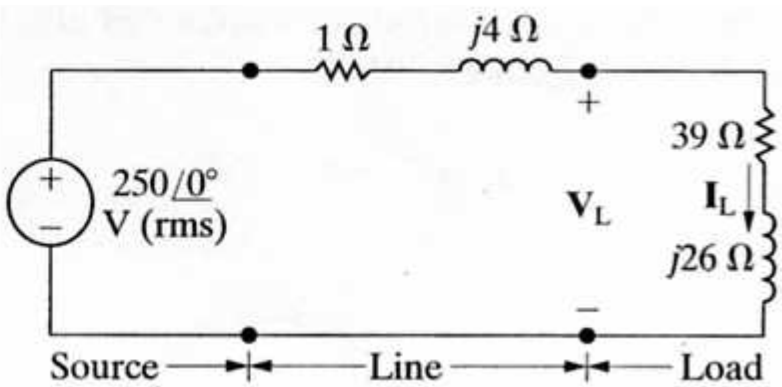
$$I_{L_{rms}} = \frac{V_{rms}}{Z} = \frac{250\angle 0}{(40+j30)}$$

$$I_{L_{rms}} = 4 - j3\ A = 5\angle -36,87\ A_{rms}$$

$$V_{2_{rms}} = I Z = (39+j26)(4-j3)$$

$$V_{L_{rms}} = 234 - j13\ V_{rms}$$

$$234,36 \angle -3,178\ V_{rms}$$



$\mathbf{I}_L = 4 - j3 = 5 \angle -36.87^\circ$ A (rms)

$\mathbf{V}_L = 234 - j13 = 234.36 \angle -3.18^\circ$ V (rms)

b) Calculate the average and reactive power delivered to the load.

$$S = \mathbf{V}_L \mathbf{I}_L^* = (234 - j13)(4 + j3) = 975 + j650 \text{ VA}$$

$P = 975$ W $\qquad Q = 650$ var

Another solution The load average power is the power absorbed by the load resistor 39 Ω

Recall the average Power for purely resistive Circuits $P = \frac{V_m^R I_m^R}{2} = V_{eff}^R I_{eff}^R$

where V_{eff}^R and I_{eff}^R Are the **rms** voltage across the resistor and the **rms** current through the resistor

$$P = V_{eff}^R I_{eff}^R = R I_{eff}^2$$

b) $S = V_{rms} I_{rms}^{*}$ Complex power

$S = (234 - j13)(4 - j3)^{*}$

$= (234 - j13)(4 + j3)$

$S = 975 + j650$ VA

$P = 975$ W
Average power

$Q = 650$ VAR
Reactive power

$$Z_L = 39 + j\,26\,\Omega$$

$$R = 39\,\Omega \qquad X_L = 26\,\Omega$$

$$I_L = 5\angle -36{,}87 \text{ Arms} \leftarrow |I_L| = 5$$

$$V_L = 234{,}36\angle -3{,}18 \text{ rms} \qquad |V_L| = 234{,}36$$

series $\Longrightarrow$ voltage divider

$$\underline{\underline{V_R}} = \frac{39}{39+j26} \times \underline{\underline{(234-j13)}}\ V_L$$

$$\underline{\underline{V_R}} = 156 - j117$$

V rms

$$= \underline{195} \angle -36.87 \rightarrow |V_R| = 195$$

$$P = \frac{|V_{eff\,r}|^2}{R} = \frac{195^2}{39} = 975\,W$$

$$P = |I_{eff}|^2 R = 5^2 \times 39 = 975\,W$$

$P = |V_{eff_R}||I_{eff}| = 195 \times 5$

$= 975W$

$Q = |V_{eff}| |I_{eff}|$

$V_L = \dfrac{j26}{j26+39} \rightarrow (234 - j13)$

$k_L = 5$

$V_{L_{eff}} = 78 + j104$

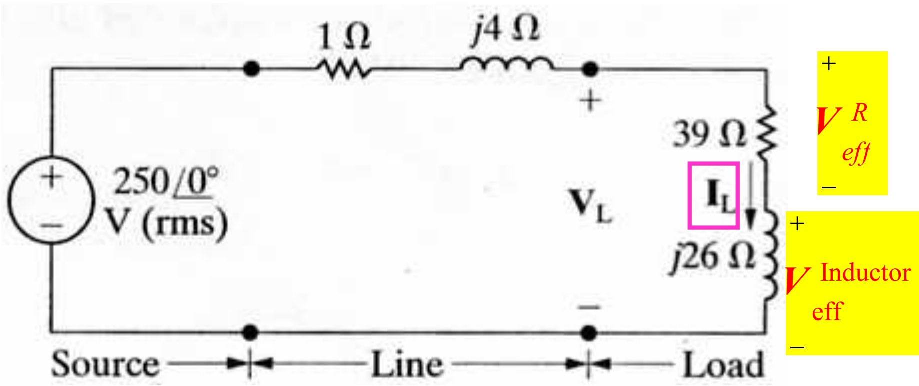
$= \underline{\underline{130}} \angle 53.13 \rightarrow |V_L| = \underline{\underline{130}}$

$Q = \dfrac{|V_{eff_L}|^2}{X} = |I_{eff_L}|^2 X$

$$Q = |V(f)| \cdot |I(f)| = 130 \times 5 = 650\,VAR$$

$$Q = \frac{|V(f)|^2}{X} = \frac{130^2}{26} = 650\,VAR$$

$$Q = |I(f)|^2 X = 5^2 \times 26 = 650\,VAR$$



$I_L = 4 - j3 = 5 \angle -36.87°$ A (rms)

$V_L = 234 - j13 = 234.36 \angle -3.18°$ V (rms)

$S = V_L I_L^* = (234 - j13)(4 + j3) = 975 + j650$ VA

$\Rightarrow P = 975$ W

$Q = 650$ var

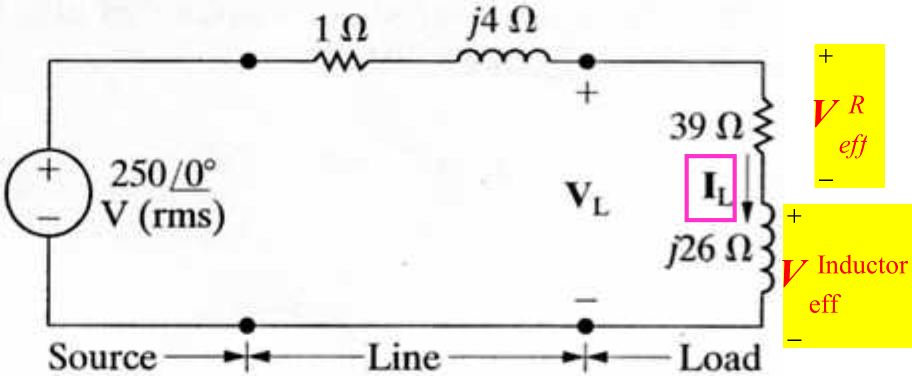
From Power for purely resistive Circuits

I_L

$$P = \frac{1}{2} V_m I_m = V_{eff} I_{eff} \Rightarrow P = |V^R_{eff}||I^R_{eff}| = V^R_{eff} I^R_{eff}$$

$$V^R_{eff} = \frac{39}{39 + j26} V_L = \frac{39}{39 + j26} 234.36 e^{-j3.18°} = \underbrace{195}_{|V^R_{eff}|} e^{-j36.87°} \Rightarrow V^R_{eff} = 195$$

$$\Rightarrow P = V^R_{eff} I^R_{eff} = (195)(5) = 975 \text{ W}$$



$\mathbf{I}_L = 4 - j3 = 5 \angle -36.87°$ A (rms)

$\mathbf{V}_L = 234 - j13 = 234.36 \angle -3.18°$ V (rms)

$S = \mathbf{V}_L \mathbf{I}_L^* = (234 - j13)(4 + j3) = 975 + j650$ VA

$\Rightarrow$ $P = 975$ W

$Q = 650$ var

From Power for purely resistive Circuits

OR $P = V^R_{eff} I^R_{eff} = (R I^R_{eff}) I^R_{eff} = R (I^R_{eff})^2 = (39)(5^2) = (39)(25) = \mathbf{975 \ \ W}$

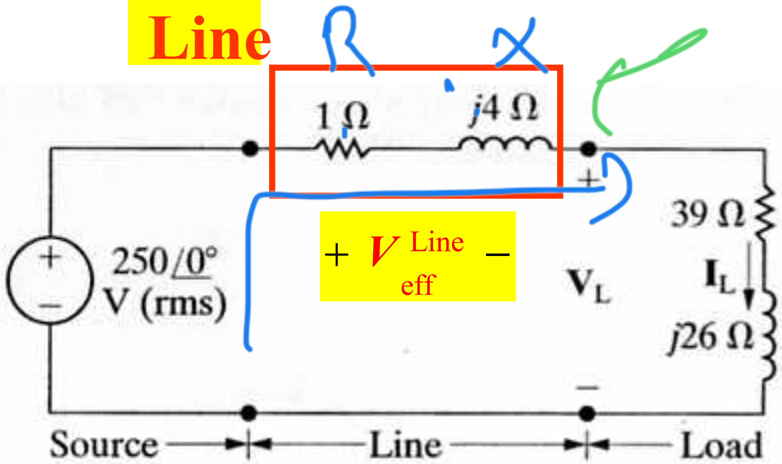
$Q = V_{eff} I_{eff} \Rightarrow Q = V^L_{eff} I^L_{eff}$ $\quad V^L_{eff} = \dfrac{j26}{39 + j26} V_L = \dfrac{j26}{39 + j26} 234.36 e^{-j3.18°} = 78 + j104 = 130 e^{j53.13°}$

$I_L = 5$

$\Rightarrow$ $V^L_{eff} = 130$ $\Rightarrow$ $Q = (130)(5) = 650$ VAR

OR $Q = X I^2_{eff} = 650$ var

Line



$\mathbf{I}_L = 4 - j3 = 5 \angle -36.87°$ A (rms)

$\mathbf{V}_L = 234 - j13 = 234.36 \angle -3.18°$ V (rms)

$S = P + jQ$

c) Calculate the average and reactive power delivered to the line.

$P = I_{\text{eff}}^2 \boldsymbol{R}$ → $P = (5)^2(1) = 25$ W,

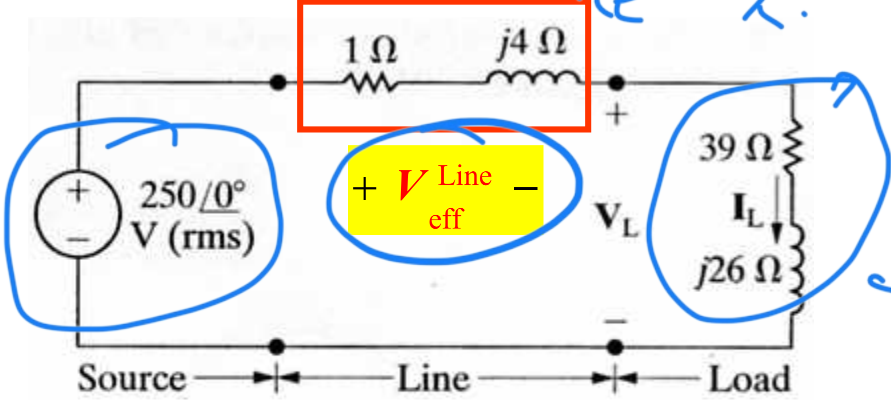
$Q = I_{\text{eff}}^2 \boldsymbol{X}$ → $Q = (5)^2(4) = 100$ VAR

$P = |I_L|^2 R \rightarrow 5^2 \times 1 = 25\text{W}$ P

$Q = |I_L|^2 X \rightarrow 5^2 \times 4 = 100$ VAR Q

$$\boxed{S = 25 + j100 \text{ VA}}$$

Line



$\mathbf{I}_L = 4 - j3 = 5\angle -36.87°$ A (rms)

$\mathbf{V}_L = 234 - j13 = 234.36\angle -3.18°$ V (rms)

$234.36\angle -3.18$

OR using complex power

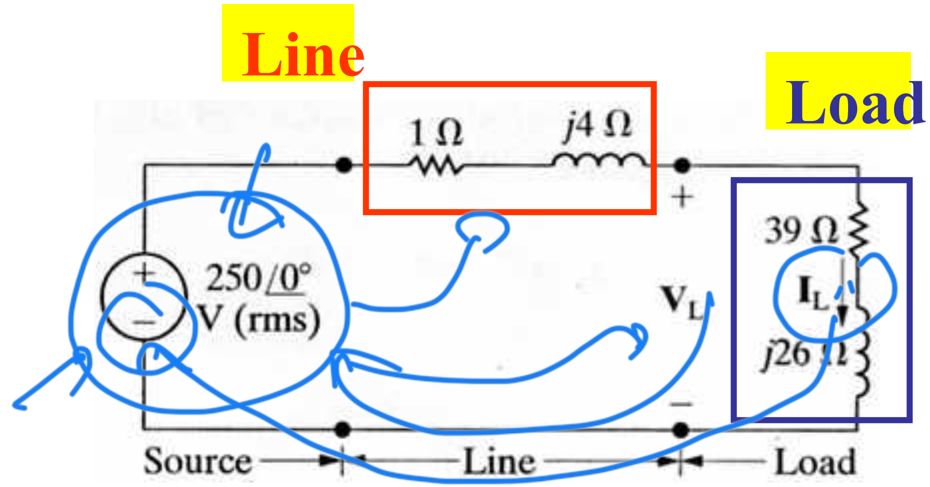
$$S_{\text{Line}} = V^{\text{Line}}_{\text{eff}} I^{*}_{\text{eff}} \qquad V^{\text{Line}}_{\text{eff}} = \frac{1+j4}{(1+j4)+(39+j26)}(250)$$

OR

$$V^{\text{Line}}_{\text{eff}} = 250 - V_L$$

$$V^{\text{Line}}_{\text{eff}} = 20.6\angle 39.1^{o} \quad \text{V rms}$$

$$S_{\text{Line}} = V^{\text{Line}}_{\text{eff}} I^{*}_{\text{eff}} = 20.6\angle 39.1^{o} \; 5\angle 36.87^{o} = 103\angle 75.97^{o} = 25 + j100 \quad \text{VA}$$



$\mathbf{I}_L = 4 - j3 = 5 \angle -36.87^\circ$ A (rms)

$\mathbf{V}_L = 234 - j13 = 234.36 \angle -3.18^\circ$ V (rms)

d) Calculate the average and reactive power supplied by the source.

$$\boldsymbol{S}_{\text{Absorb}} = \boldsymbol{S}_{Line} + \boldsymbol{S}_{Load} = \overbrace{(25 + j\,100)}^{\text{From part (c)}} + \overbrace{(975 + j\,650)}^{\text{From part (b)}} = (25 + 975) + j\,(100 + 650) = 1000 + j\,750 \quad \text{VA}$$

$$\Rightarrow \boldsymbol{S}_{\text{Supply}} = -\boldsymbol{S}_{\text{Absorb}} = -(1000 + j\,750) \quad \text{VA}$$

OR

$$\boldsymbol{S}_{\text{Supply}} = -250 \angle 0^\circ \, (\boldsymbol{I}_L^{*}) = -250 \angle 0^\circ \; 5 \angle 36.87^\circ = \mathbf{-1250} \angle 36.87^\circ \quad \text{VA}$$

$$= -1000 - j\,750 \quad \text{VA}$$

$S_{\text{مصدر}} = S_{line} + S_{load}$

$= \underline{\underline{25 + j100}} + \underline{97S + j650}$

$S = -(1000 + j750)$

absorbed + يستهلك

supplied − ينتج

$S = -1000 - j750$ VA

$$S = -V_s I_s^*$$

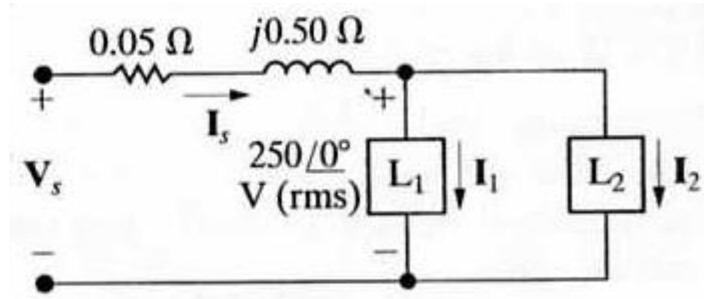
$$S = -[250\angle 0][4-j3]^*$$

$$S = -\underline{\underline{250}} \times (4+j3)$$

$$\boxed{S = -1000 - j750 \text{ VA}}$$

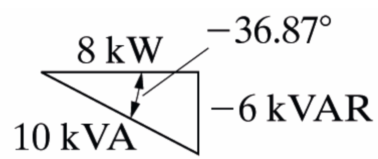
Example 10.6 Calculating Power in Parallel Loads

The two loads in the circuit shown can be described as follows: Load 1 absorbs an average power of 8 kW at a leading power factor of 0.8. Load 2 absorbs 20 kVA at a lagging power factor of 0.6.

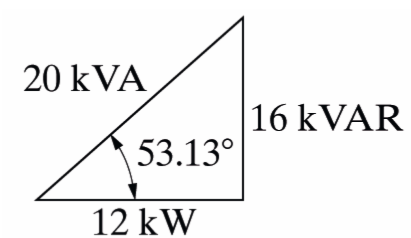


a) Determine the power factor of the two loads in parallel.

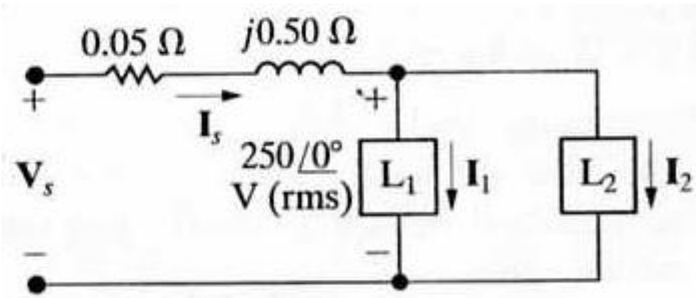
$\mathbf{I}_s = \mathbf{I}_1 + \mathbf{I}_2$ $\quad S = (250)\mathbf{I}_s^* \quad = (250)(\mathbf{I}_1 + \mathbf{I}_2)^* \quad = (250)\mathbf{I}_1^* + (250)\mathbf{I}_2^* \quad = S_1 + S_2$



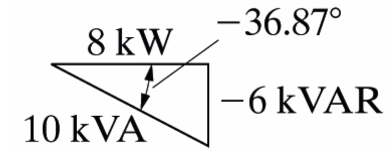
$$S_1 = 8000 - j\frac{8000(.6)}{(.8)} = 8000 - j6000 \text{ VA}$$



$$S_2 = 20{,}000(.6) + j20{,}000(.8) = 12{,}000 + j16{,}000 \text{ VA}$$

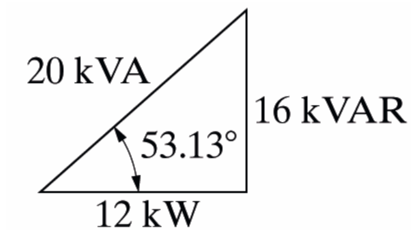


$$\mathbf{I}_s = \mathbf{I}_1 + \mathbf{I}_2$$



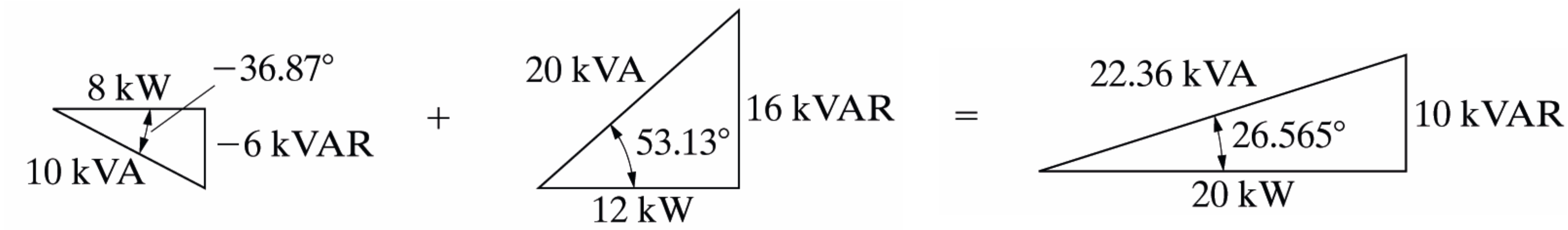
$$S_1 = 8000 - j\frac{8000(.6)}{(.8)}$$

$$= 8000 - j6000 \text{ VA}$$



$$S_2 = 20{,}000(.6) + j20{,}000(.8)$$

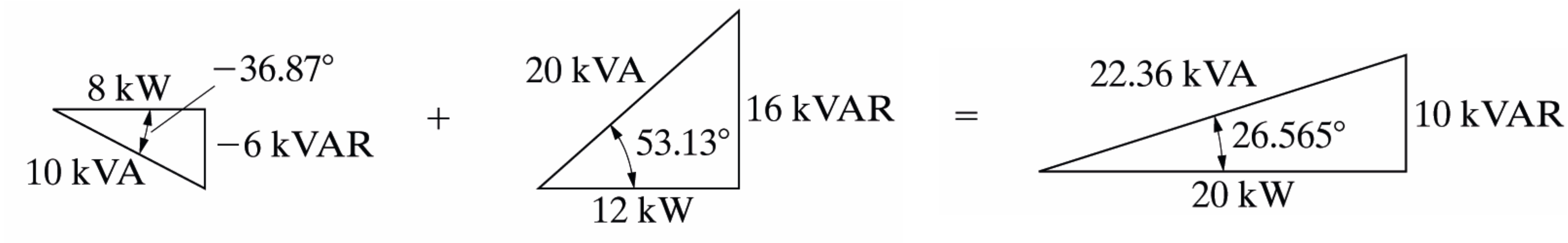
$$= 12{,}000 + j16{,}000 \text{ VA}$$



$$S = 20{,}000 + j10{,}000 \text{ VA}$$

$$\mathbf{I}_s^* = \frac{20{,}000 + j10{,}000}{250} = 80 + j40 \text{ A}$$

$$\mathbf{I}_s = 80 - j40 = 89.44 \angle -26.57^\circ \text{ A}$$

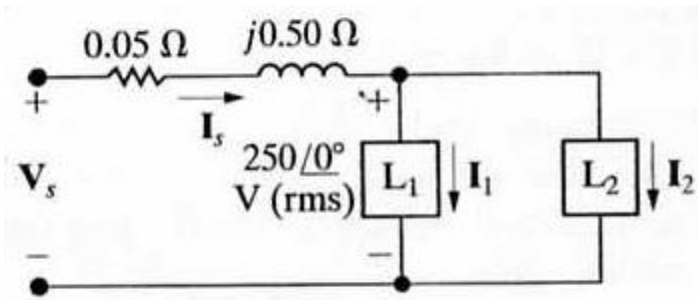


$$S = 20{,}000 + j10{,}000 \text{ VA}$$

$$\mathbf{I}_s^* = \frac{20{,}000 + j10{,}000}{250} = 80 + j40 \text{ A} \qquad \mathbf{I}_s = 80 - j40 = 89.44 \angle -26.57^\circ \text{ A}$$

$$\textbf{pf} = \cos(\theta_v - \theta_i) \qquad \text{pf} = \cos(0 + 26.57^\circ) = 0.8944 \text{ lagging}$$

The power factor of the two loads in parallel is lagging because the net reactive power is positive.



b) Determine the apparent power required to supply the loads, the magnitude of the current, $\mathbf{I}_s$, and the average power loss in the transmission line.

$$\mathbf{I}_s = 80 - j40 = 89.44 \angle -26.57^\circ \text{ A}$$

$\boldsymbol{S}_1 = 8000 - j6000 \quad \text{VA}$ $\boldsymbol{S}_2 = 12000 + j16000 \quad \text{VA}$ $\boldsymbol{S} = 20000 + j10000 \quad \text{VA}$

The apparent power which must be supplied to these loads is

$$|\boldsymbol{S}| = |20000 + j10000| \quad \text{VA} \quad = 22.36 \quad \text{kVA}$$

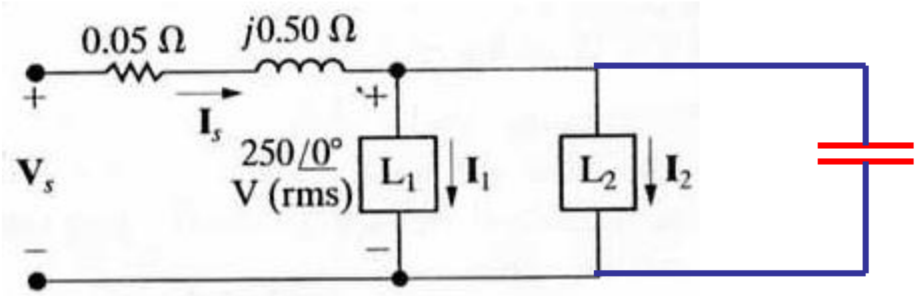
The magnitude of the current that supplies this apparent power is

$$|\mathbf{I}_s| = |80 - j40| \quad = 89.44 \text{ A}$$

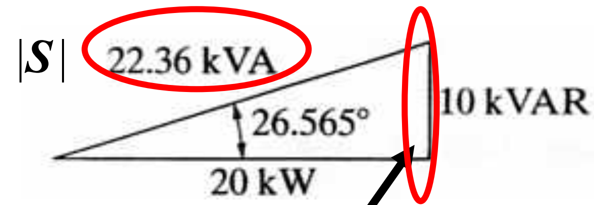
The average power lost in the line results from the current flowing through the line resistance

$$P_{\text{line}} = |\mathbf{I}_s|^2 R = (89.44)^2(0.05) = 400 \text{ W}$$

Note that the power supplied totals 20,000 + 400 = 20,400 W, even though the loads require a total of only 20,000 W.



C ?



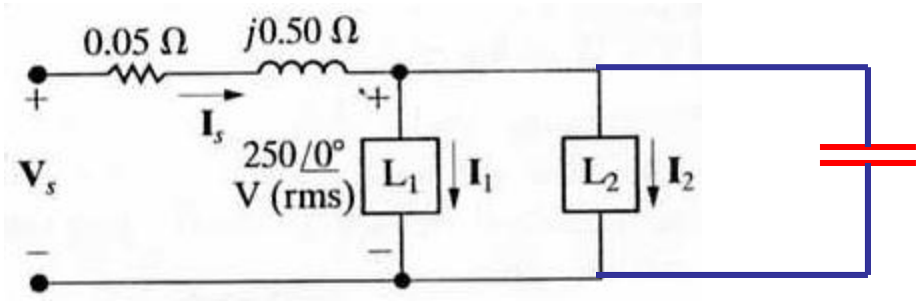
c) Given that the frequency of the source is 60 Hz, compute the value of the capacitor that would correct the power factor to 1 if placed in parallel with the two loads. Recompute the values in (b) for the load with the corrected power factor.

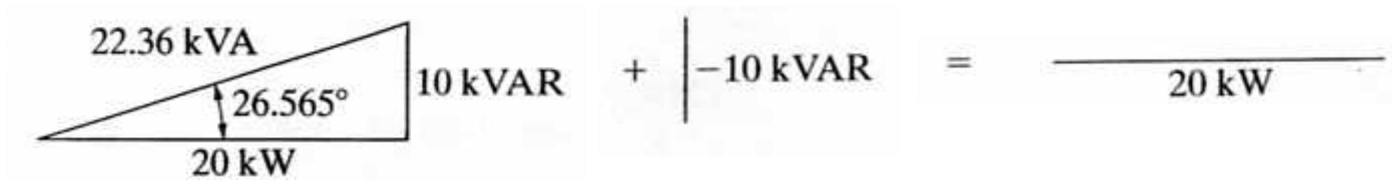
Will cancel this

As we can see from the power triangle We can correct the power factor to 1 if we place a capacitor in parallel with the existing load

the capacitive reactance $X \doteq \frac{|V_{eff}|^2}{Q} = \frac{(250)^2}{-10{,}000} = -6.25\ \Omega.$

Recall that $X = -\frac{1}{\omega C}$ $\omega = 2\pi(60) = 376.99$ rad/s $C = \frac{-1}{\omega X} = \frac{-1}{(376.99)(-6.25)} = 424.4\ \mu F.$



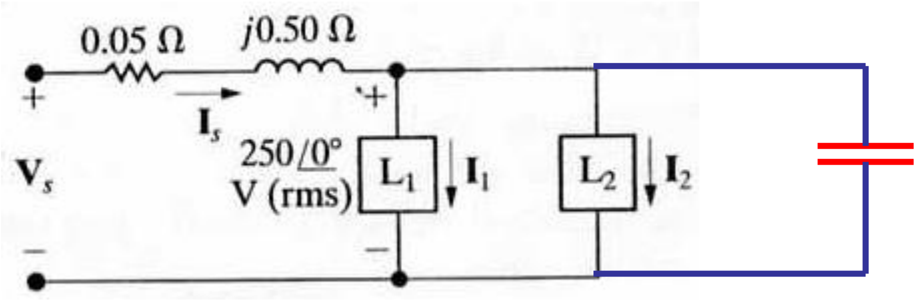


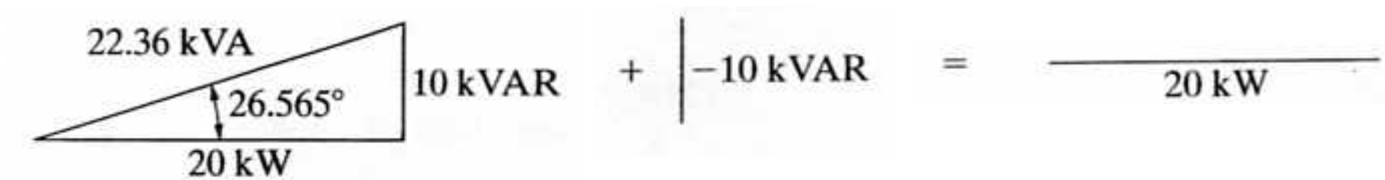
When the power factor is 1, the apparent power and the average power are the same.

$$|S| = P = 20 \text{ kVA}$$

The magnitude of the current that supplies this apparent power is

$$|\mathbf{I}_s| = \frac{20{,}000}{250} = 80 \text{ A}$$



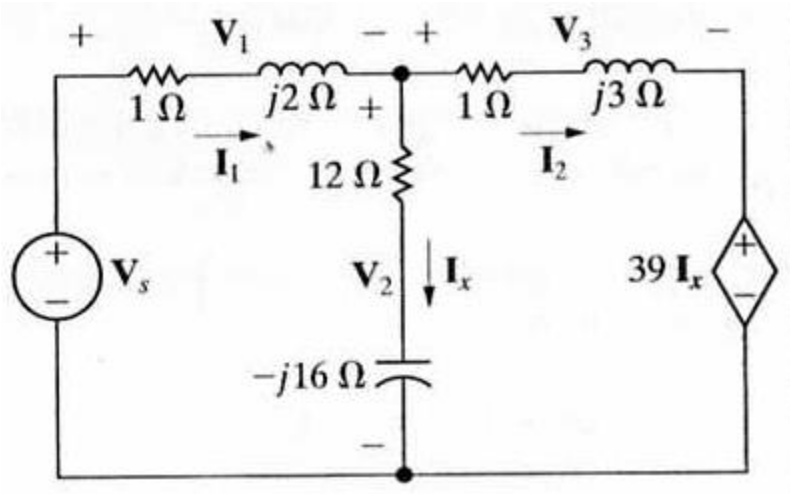


The average power lost in the line is thus reduced to $P_{\text{line}} = |\mathbf{I}_s|^2 R = (80)^2(0.05) = 320$ W

Now, the power supplied totals 20,000 + 320 = 20,320 W

The addition of the capacitor has reduced the line loss from **400 W** to **320 W**

Example 10.7



$\mathbf{V}_s = 150\angle 0^\circ$ V

$\mathbf{V}_1 = (78 - j104)$ V $\quad \mathbf{I}_1 = (-26 - j52)$ A

$\mathbf{V}_2 = (72 + j104)$ V $\quad \mathbf{I}_x = (-2 + j6)$ A

$\mathbf{V}_3 = (150 - j130)$ V $\quad \mathbf{I}_2 = (-24 - j58)$ A

a) Calculate the total average and reactive power delivered to each impedance in the circuit

$$\mathbf{S}_1 = \frac{1}{2}\mathbf{V}_1\mathbf{I}_1^* = \frac{1}{2}(78 - j104)(-26 + j52) = 1690 + j3380 \text{ VA} = P_1 + jQ_1$$

$$P_1 = 1690 \text{ W} \quad \text{and} \quad Q_1 = 3380 \text{ VAR}$$

Another solution

$$P = \frac{1}{2} R |\mathbf{I_R}|^2 = \frac{1}{2}\frac{|\mathbf{V_R}|^2}{R} \Rightarrow P_1 = \frac{1}{2}(1)|\mathbf{I_1}|^2 = \frac{1}{2}(1)\left(\sqrt{(-26)^2 + (-52)^2}\right)^2 = 1690 \text{ W}$$

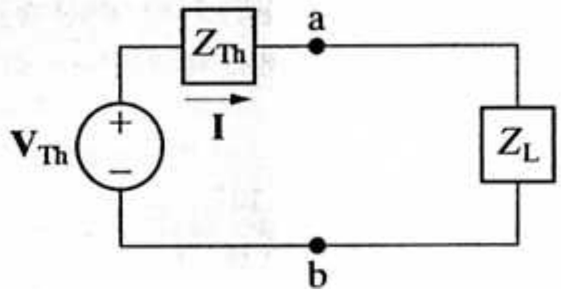
OR $$\mathbf{V_R} = \frac{1}{1+j2}(\mathbf{V_1}) \Rightarrow P_1 = \frac{1}{2}\frac{|\mathbf{V_R}|^2}{1} = 1690 \text{ W}$$

Similarly

$$Q = \frac{1}{2} X |\mathbf{I_X}|^2 = \frac{1}{2}\frac{|\mathbf{V_X}|^2}{X} \Rightarrow Q_1 = \frac{1}{2}(2)|\mathbf{I_1}|^2 = \frac{1}{2}(2)\left(\sqrt{(-26)^2 + (-52)^2}\right)^2 = 3380 \text{ VAR}$$

OR $$\mathbf{V_X} = \frac{j2}{1+j2}(\mathbf{V_1}) \Rightarrow Q_1 = \frac{1}{2}\frac{|\mathbf{V_X}|^2}{2} = 3380 \text{ VAR}$$

10.5 Maximum Power Transfer



$Z_{Th} = R_{Th} + jX_{Th}$ $\qquad$ $Z_L = R_L + jX_L$

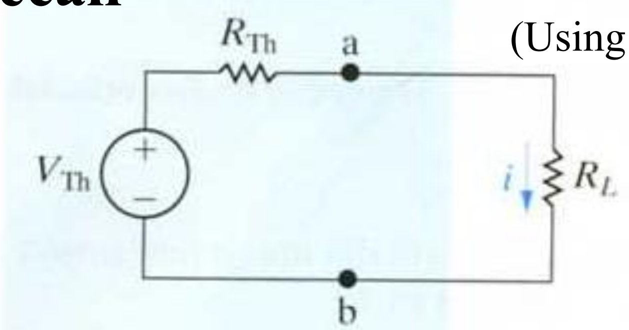
What is the value of Z_L that will absorb maximum power

For maximum power transfer :

$$Z_L = R_{TH} - jX_{TH} \quad \Rightarrow Z_L = Z_{TH}^*$$

Recall

(Using rms values)



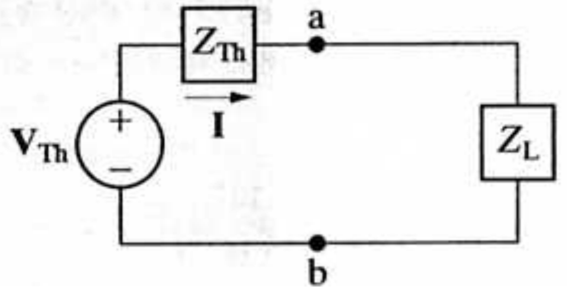
$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(R_{Th} + R_L)^2 - R_L \cdot 2(R_{Th} + R_L)}{(R_{Th} + R_L)^4} \right] = 0$$

$$R_L = R_{Th}.$$

Similarly

$$Z_{Th} = R_{Th} + jX_{Th}$$



$$Z_L = R_L + jX_L$$

load current **I** is $$\mathbf{I} = \frac{\mathbf{V}_{Th}}{(R_{Th} + R_L) + j(X_{Th} + X_L)}$$

The average power delivered to the load is $P = |\mathbf{I}|^2 R_L$

$$P = \frac{|\mathbf{V}_{Th}|^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

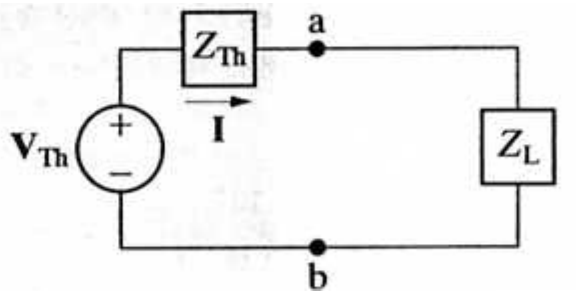
$$\frac{\partial P}{\partial X_L} = \frac{-|\mathbf{V}_{Th}|^2 2R_L(X_L + X_{Th})}{[(R_L + R_{Th})^2 + (X_L + X_{Th})^2]^2}$$

$\partial P/\partial X_L$ is zero when

$$X_L = -X_{Th}$$

Similarly

$$Z_{Th} = R_{Th} + jX_{Th}$$



$$Z_L = R_L + jX_L$$

$\partial P/\partial X_L$ is zero when $X_L = -X_{Th}$

$$\frac{\partial P}{\partial R_L} = \frac{|V_{Th}|^2[(R_L + R_{Th})^2 + (X_L + X_{Th})^2 - 2R_L(R_L + R_{Th})]}{[(R_L + R_{Th})^2 + (X_L + X_{Th})^2]^2}$$

$\partial P/\partial R_L$ is zero when $R_L = \sqrt{R_{Th}^2 + (X_L + X_{Th})^2}$

=0 (since $X_L = -X_{th}$)

$$R_L = R_{TH}$$

$$\boxed{Z_L = Z_{Th}^*}$$

$$Z_L = R_{TH} - jX_{TH}$$

Example 10.11 Finding Max Power Transfer in a circuit with an Ideal Transfo.

Finding Max Power Transfer in a circuit.

a) The value of R_L

b) What is the maximum average power delivered to R_L

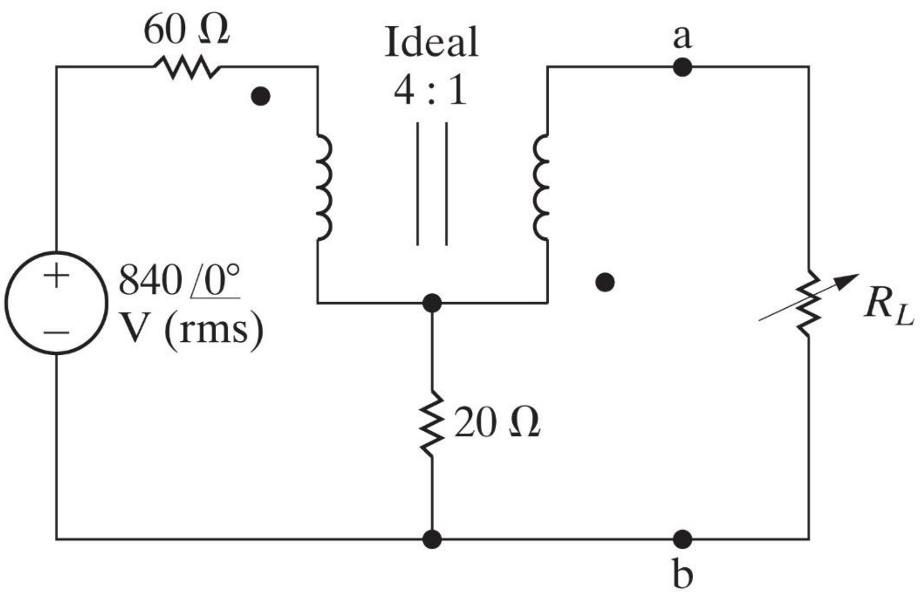


Figure: 10-24Ex10.11

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Example 10.11 Finding Max Power Transfer in a circuit with an Ideal Transfo.

a)The value of $R_L = \dfrac{V_{Th}}{I_{SC}} = \dfrac{-210}{-6} = 35\Omega$

b)The maximum average power delivered to R_L is $P_{max} = \left(\dfrac{-210}{70}\right)^2 \times 35 = 315W$

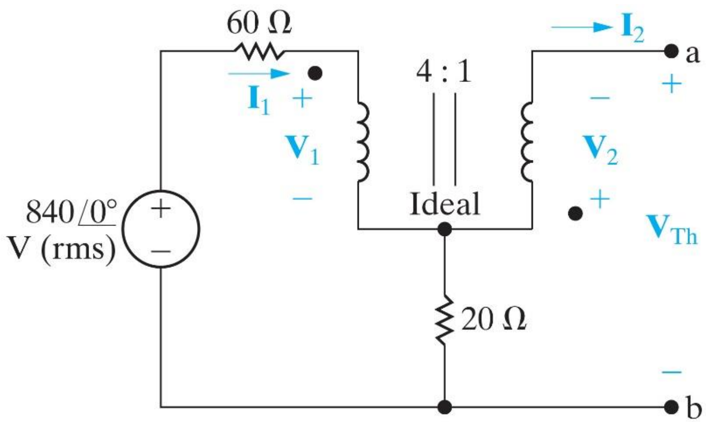

Figure: 10-25Ex10.11

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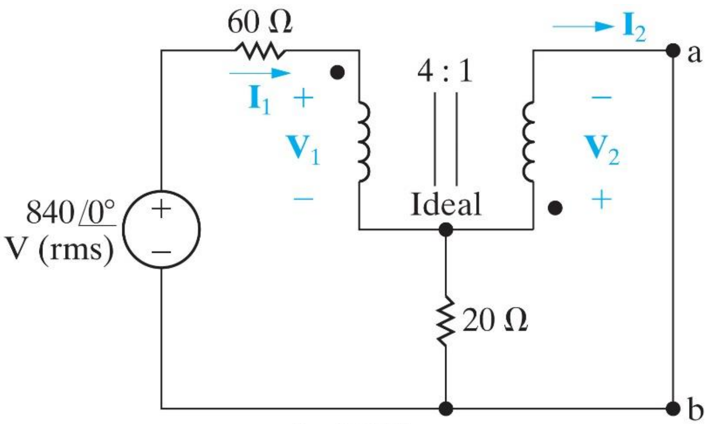

Figure: 10-26Ex10.11

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