Tutorial chapter 1

1.2 Points $A(4, -6, 2)$, $B(-2, 0, 3)$, and $C(10, 1, -7)$ form a triangle. Show that $\mathbf{r}_{AB} + \mathbf{r}_{BC} + \mathbf{r}_{CA} = 0$.

zero

$$r_{AB} = r_B - r_A = (-2,0,3) - (4,-6,2) = (-6,6,1)$$

$$r_{BC} = r_C - r_B = (10,1,-7) - (-2,0,3) = (12,1,-10)$$

$$r_{CA} = r_A - r_C = (4,-6,2) - (10,1,-7) = (-6,-7,9)$$

$$r_{AB} + r_{BC} + r_{CA} = (0,0,0) = \mathbf{0}$$

1.3 If $\mathbf{A} = 4\mathbf{a}_x - 2\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 12\mathbf{a}_x + 18\mathbf{a}_y - 8\mathbf{a}_z$, determine:

(a) $\mathbf{A} - 3\mathbf{B}$

(b) $(2\mathbf{A} + 5\mathbf{B})/|\mathbf{B}|$

(c) $\mathbf{a}_x \times \mathbf{A}$

(d) $(\mathbf{B} \times \mathbf{a}_x) \cdot \mathbf{a}_y$

(a)

$$\boldsymbol{A} - 3\boldsymbol{B} = (4,-2,6) - 3(12,18,-8) = (4,-2,6) - (36,54,-24)$$

$$= \underline{\underline{(-32,-56,-30)}}$$

(b)

$$2\boldsymbol{A} + 5\boldsymbol{B} = 2(4,-2,6) + 5(12,18,-8) = (68,86,-28)$$

$$|\boldsymbol{B}| = \sqrt{12^2 + 18^2 + 8^2} = \sqrt{532} = 23.065$$

$$(2\boldsymbol{A} + 5\boldsymbol{B})/|\boldsymbol{B}| = (68,86,-28)/23.065 = 2.948\boldsymbol{a}_x + 3.728\boldsymbol{a}_y - 1.214\boldsymbol{a}_z$$

(c)

$$\boldsymbol{a}_x \times \boldsymbol{A} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -2 & 6 \end{vmatrix} = \underline{\underline{-6\boldsymbol{a}_y - 2\boldsymbol{a}_z}}$$

(d)

$$\boldsymbol{B} \times \boldsymbol{a}_x = \begin{vmatrix} 12 & 18 & -8 \\ 1 & 0 & 0 \end{vmatrix} = -8\boldsymbol{a}_y - 18\boldsymbol{a}_z$$

$$(\boldsymbol{B} \times \boldsymbol{a}_x) \bullet \boldsymbol{a}_y = -8$$

1.2

$A(4, -6, 2)$

$B(-2, 0, 3)$

$C(10, 1, -7)$

$$\vec{r_{AB}} = B - A = \langle -6, 6, 1 \rangle$$

$$+\ r_{BC} = C - B = \langle 12, 1, -10 \rangle\ +$$

$$r_{CA} = A - C = \langle -6, -7, 9 \rangle$$

$$\vec{r_{AB}} + \vec{r_{BC}} + \vec{r_{CA}} = \langle 0, 0, 0 \rangle$$

$$= \vec{0}$$

1.3

$$\vec{A} = 4a_x - 2a_y + 6a_z$$

$$\vec{B} = 12a_x + 18a_y - 8a_z$$

A) $A - 3B = 4a_x - 2a_y + 6a_z$

$$+ [-36a_x - 54a_y + 24a_z]$$

$$\boxed{A - 3B = -32a_x - 56a_y + 30a_z}$$

B) $|2A + 5B| / |B|$

$$|B| = \sqrt{12^2 + 18^2 + 8^2} = 2\sqrt{133}$$

$$2A + 5B = 8a_x - 4a_y + 12a_z$$

$$+ 60a_x + 90a_y - 40a_z$$

$$\frac{2A + 5B}{|B|} = \frac{68}{2\sqrt{133}} a_x + \frac{86}{2\sqrt{133}} a_y - \frac{28}{2\sqrt{133}} a_z$$

e) $\vec{ax} \times A$

$\vec{ax} = \langle 1, 0, 0 \rangle$

$A = \langle 4, -2, 6 \rangle$

$$\vec{ax} \times \vec{A} = \begin{vmatrix} a_x & a_y & a_z \\ 1 & 0 & 0 \\ 4 & -2 & 6 \end{vmatrix}$$

$\cancel{0\,\vec{ax}} - a_y\,6 + -2\,a_z$

$ax \times A = -6a_y - 2a_z$

d) $(B \times a_x) \cdot \underline{\underline{a_y}}$

$\vec{a_x} = \langle 1, 0, 0 \rangle$

$\vec{a_y} = \langle 0, 1, 0 \rangle$

$\vec{B} = \langle 12, 18, -8 \rangle$

$$B \times a_x = \begin{array}{ccc} + & - & + \\ \end{array} \begin{vmatrix} a_x & a_y & a_z \\ 12 & 18 & -8 \\ 1 & 0 & 0 \end{vmatrix}$$

$$0\,\vec{a_x} - 8\,a_y + -18\,a_z$$

$$\vec{B} \times \vec{a_x} = \begin{matrix} 0\,a_x & -8\,a_y & +18\,a_z \\ 0\,a_x & +1\,a_y & +0\,a_z \end{matrix}$$

$$(\vec{B} \times \vec{a_x}) \cdot \vec{a_y} = 0 + -8 + 0$$

$$= \boxed{-8} \checkmark$$

$$B = 12a_x + 18a_y - 8a_z$$

$$a_x = 1a_x + 0a_y + 0a_z$$

$$a_y = 0a_x + 1a_y + 0a_z$$

$$(B \times a_x) \cdot a_y = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{vmatrix} 12 & 18 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$+12(0) - 18(0) + -8(1)$$

$$\boxed{-8}$$

1.5 Let $\mathbf{A} = -2\mathbf{a}_x + 5\mathbf{a}_y + \mathbf{a}_z$, $\mathbf{B} = \mathbf{a}_x + 3\mathbf{a}_z$, and $\mathbf{C} = 4\mathbf{a}_x - 6\mathbf{a}_y + 10\mathbf{a}_z$.

(a) Determine $\mathbf{A} - \mathbf{B} + \mathbf{C}$

(b) Find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

(c) Calculate the angle between $\mathbf{A}$ and $\mathbf{B}$

(a) $\boldsymbol{A} - \boldsymbol{B} + \boldsymbol{C} = (-2,5,1) + (-1,0,-3) + (4,-6,10) = \underline{\underline{(1,-1,8)}}$

(b) $$\boldsymbol{B} \times \boldsymbol{C} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -6 & 10 \end{vmatrix} = (18,2,-6)$$

$$\boldsymbol{A} \bullet (\boldsymbol{B} \times \boldsymbol{C}) = (-2,5,1) \bullet (18,2,-6) = -36 + 10 - 6 = \underline{\underline{-32}}$$

(c) $$\cos\theta_{AB} = \frac{\boldsymbol{A} \bullet \boldsymbol{B}}{AB} = \frac{-2+0+3}{\sqrt{4+25+1}\sqrt{1+9}} = 0.05773 \quad \rightarrow \quad \theta_{AB} = \underline{\underline{86.69^\circ}}$$

1.12 If $\mathbf{A} = 4\mathbf{a}_x - 6\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + 5\mathbf{a}_z$, find:

(a) $\mathbf{A} \cdot \mathbf{B} + 2|\mathbf{B}|^2$

(b) a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$

$$\boldsymbol{A} \square \boldsymbol{B} = (4,-6,1) \square (2,0,5) = 8 - 0 + 5 = 13$$

(a) $$|\boldsymbol{B}|^2 = 2^2 + 5^2 = 29$$

$$\boldsymbol{A} \square \boldsymbol{B} + 2|\boldsymbol{B}|^2 = 13 + 2 \times 29 = \underline{\underline{71}}$$

(b)

$$\boldsymbol{a}_\perp = \pm \frac{\boldsymbol{A} \times \boldsymbol{B}}{|\boldsymbol{A} \times \boldsymbol{B}|}$$

Let $$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} 4 & -6 & 1 \\ 2 & 0 & 5 \end{vmatrix} = (-30,-18,12)$$

$$\boldsymbol{a}_\perp = \pm \frac{\boldsymbol{C}}{|\boldsymbol{C}|} = \pm \frac{(-30,-18,12)}{\sqrt{30^2 + 18^2 + 12^2}} = \underline{\underline{\pm(-0.8111\boldsymbol{a}_x - 0.4867\boldsymbol{a}_y + 0.3244\boldsymbol{a}_z)}}$$

1.13 Determine the dot product, cross product, and angle between

$$\mathbf{P} = 2\mathbf{a}_x - 6\mathbf{a}_y + 5\mathbf{a}_z \quad \text{and} \quad \mathbf{Q} = 3\mathbf{a}_y + \mathbf{a}_z$$

5-6

1.5

$$A = -2a_x + 5a_y + 1a_z$$

$$B = 1a_x + 0a_y + 3a_z \quad -2$$

$$C = 4a_x - 6a_y + 10a_z$$

a) $A - B + C = 1\vec{a_x} - 1\vec{a_y} + 8\vec{a_z}$

$$= \langle 1, -1, 8 \rangle$$

B) $A \cdot (B \times C)$

$$A \cdot (B \times C) = \begin{vmatrix} -2 & 5 & 1 \\ 1 & 0 & 3 \\ 4 & -6 & 10 \end{vmatrix}$$

$$\underbrace{-2(18)} - 5(\underbrace{10 - 12}_{-2}) + 1(-6)$$

$$-36 + 10 - 6 = \boxed{-32}$$

c) angle between A and B

$$A = \langle \boxed{-2,} \boxed{5,} \boxed{1} \rangle$$

$$B = \langle \boxed{1,} \boxed{0,} \boxed{3} \rangle$$

$$A \cdot B = |A|\,|B| \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{A \cdot B}{|A|\,|B|}\right)$$

$$\vec{A} \cdot \vec{B} = -2 + 0 + 3 = 1$$

$$|A| = \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}$$

$$|B| = \sqrt{1^2 + 0^2 + 3^2} = \sqrt{10}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{30}\,\sqrt{10}}\right) = \underline{\underline{86{,}69^\circ}}$$

1.12

$$A = 4a_x + 6a_y + 1a_z$$
$$B = 2a_x + 0a_y + 5a_z$$

a) $A \cdot B + 2|B|^2$

$\rightarrow A \cdot B = 8 + 0 + 5 = 13$

$$|B| = \sqrt{2^2 + 0^2 + 5^2} = \sqrt{29}$$

$$|B|^2 = \left(\sqrt{29}\right)^{\cancel{2}} = 29$$

$$13 + 2 \cdot 29$$

$$13 + 58 = \boxed{71}$$

B) a unit vector perpendicular to both A and B

perpendicular both A and B

$\Longrightarrow \vec{A} \times \vec{B}$ or $\vec{B} \times \vec{A}$

$\vec{A} \times \vec{B}$

$a_x(-30) - a_y(20 - 2) + a_z(12)$

$\vec{A} \times \vec{B} = -30a_x - 18a_y + 12a_z$

$|A \times B| = \sqrt{30^2 + 18^2 + 12^2}$

$= 6\sqrt{38}$

$$\text{Unit vector} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

Unit vector

$$= \mp \left[\frac{-30 a_x}{6\sqrt{38}} - \frac{18 a_y}{6\sqrt{38}} + \frac{12 a_z}{6\sqrt{38}} \right]$$

$$+ \vec{A} \times \vec{B} = - \vec{B} \times \vec{A}$$

$$\boldsymbol{P} \cdot \boldsymbol{Q} = (2,-6,5) \cdot (0,3,1) = 0 - 18 + 5 = \underline{-13}$$

$$\boldsymbol{P} \times \boldsymbol{Q} = \begin{vmatrix} 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix} = \underline{\underline{-21\boldsymbol{a}_x - 2\boldsymbol{a}_y + 6\boldsymbol{a}_z}}$$

$$\cos\theta_{PQ} = \frac{\boldsymbol{P} \cdot \boldsymbol{Q}}{PQ} = \frac{-13}{\sqrt{10}\sqrt{65}} = -0.51 \longrightarrow \theta_{PQ} = \underline{\underline{120.66^{o}}}$$

1.14 Prove that vectors $\mathbf{P} = 2\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z$ and $\mathbf{Q} = 5\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z$ $\boxed{\text{are orthogonal vectors.}}$

P and **Q** are orthogonal if the angle between them is 90°. Hence

$$\boldsymbol{P} \bullet \boldsymbol{Q} = PQ\cos\theta = 0$$

$$\boldsymbol{P} \bullet \boldsymbol{Q} = (2, 4, -6) \bullet (5, 2, 3) = 10 + 8 - 18 = 0$$

showing that they are perpendicular or orthogonal.

perpendicular

$$\underline{\underline{\vec{P} \cdot \vec{Q} = 0}}$$

1.13

$$P = 2a_x - 6a_y + 5a_z$$
$$Q = 0a_x + 3a_y + 1a_z$$

$$\text{Dot} \Rightarrow \vec{P} \cdot \vec{Q} = 0 + -18 + 5 = -13$$

$$\text{Cross} \Rightarrow \vec{P} \times \vec{Q} = \begin{vmatrix} a_x & a_y & a_z \\ 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix}$$

$$a_x(-6 - 15) - a_y(2) + a_z(6)$$

$$-21a_x - 2a_y + 6a_z$$

$$|P| = \sqrt{2^2 + 6^2 + 5^2} = \sqrt{65}$$
$$|Q| = \sqrt{0^2 + 3^2 + 1^2} = \sqrt{10}$$

$$\theta = \cos^{-1}\left(\frac{P \cdot Q}{|P|\,|Q|}\right) = \cos^{-1}\left(\frac{-13}{\sqrt{10}\sqrt{65}}\right) \Rightarrow$$

$$\Rightarrow \theta = 120{,}657^\circ$$

1.14

$P = \boxed{2a_x} + \boxed{4a_y} - \boxed{6a_z}$

$Q = \boxed{5a_x} + \boxed{2a_y} + \boxed{3a_z}$

$\vec{P} \cdot \vec{Q} = \underbrace{10 + 8}_{18} + -18$

$18 - 18 = 0$

$\Rightarrow \vec{P} \cdot \vec{Q} = 0$

$\Rightarrow$ **perpendicular**

orthogonal

EXAMPLE 1.7 Show that points $P_1(5, 2, -4)$, $P_2(1, 1, 2)$, and $P_3(-3, 0, 8)$ all lie on a straight line. Determine the shortest distance between the line and point $P_4(3, -1, 0)$.

Solution:

The distance vector $\mathbf{r}_{P_1P_2}$ is given by

$$\begin{aligned}\mathbf{r}_{P_1P_2} = \mathbf{r}_{P_2} - \mathbf{r}_{P_1} &= (1, 1, 2) - (5, 2, -4)\\ &= (-4, -1, 6)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{r}_{P_1P_3} = \mathbf{r}_{P_3} - \mathbf{r}_{P_1} &= (-3, 0, 8) - (5, 2, -4)\\ &= (-8, -2, 12)\\ \mathbf{r}_{P_1P_4} = \mathbf{r}_{P_4} - \mathbf{r}_{P_1} &= (3, -1, 0) - (5, 2, -4)\\ &= (-2, -3, 4)\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{P_1P_2} \times \mathbf{r}_{P_1P_3} &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ -4 & -1 & 6 \\ -8 & -2 & 12 \end{vmatrix}\\ &= (0, 0, 0)\end{aligned}$$

showing that the angle between $\mathbf{r}_{P_1P_2}$ and $\mathbf{r}_{P_1P_3}$ is zero ($\sin\theta = 0$). This implies that P_1, P_2, and P_3 lie on a straight line.

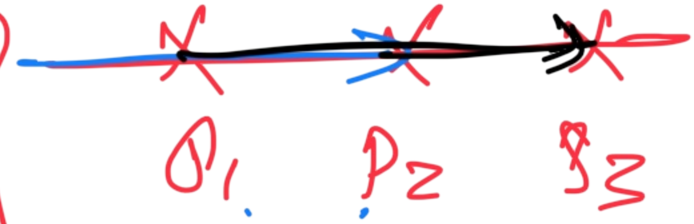
Alternatively, the vector equation of the straight line is easily determined from Figure 1.12(a). For any point P on the line joining P_1 and P_2

$$\mathbf{r}_{P_1P} = \lambda \mathbf{r}_{P_1P_2}$$

where λ is a constant. Hence the position vector $\mathbf{r}_P$ of the point P must satisfy

1.7

$$P_1\,(5,\ 2,\ -4)$$
$$P_2\,(1,\ 1,\ 2)$$
$$P_3\,(-3,\ 0,\ 8)$$



نفس متجه

$$r_{P_1P_2} = P_2 - P_1 = \langle -4,\ -1,\ 6 \rangle$$

$$r_{P_{23}} = P_3 - P_2 = \langle -4,\ -1,\ 6 \rangle$$

$$r_{P_1P_3} = P_3 - P_1 = \langle -8,\ -2,\ 12 \rangle$$

$$r_{P_1P_2} = r_{P_{23}}$$

$$r_{P_1P_3} = 2\,r_{P_1P_2} \ \text{or}\ 2\,r_{P_2P_3}$$

⇐ كيف نثبت أن النقاط تقع على خط مستقيم واحد ؟

* قبل أي شي أوجد متجهات بين هذه النقاط

$\boxed{1}$

$$r_1 = r_2 = r_3$$

$$r_1 = k\, r_2$$

$$r_2 = \lambda\, r_3$$

متجهات متوازية parallel

$\boxed{2}$

$$\vec{r_1} \times \vec{r_2} = \langle 0, 0, 0 \rangle$$

$\Rightarrow$ parallel

تقع على خط مستقيم واحد

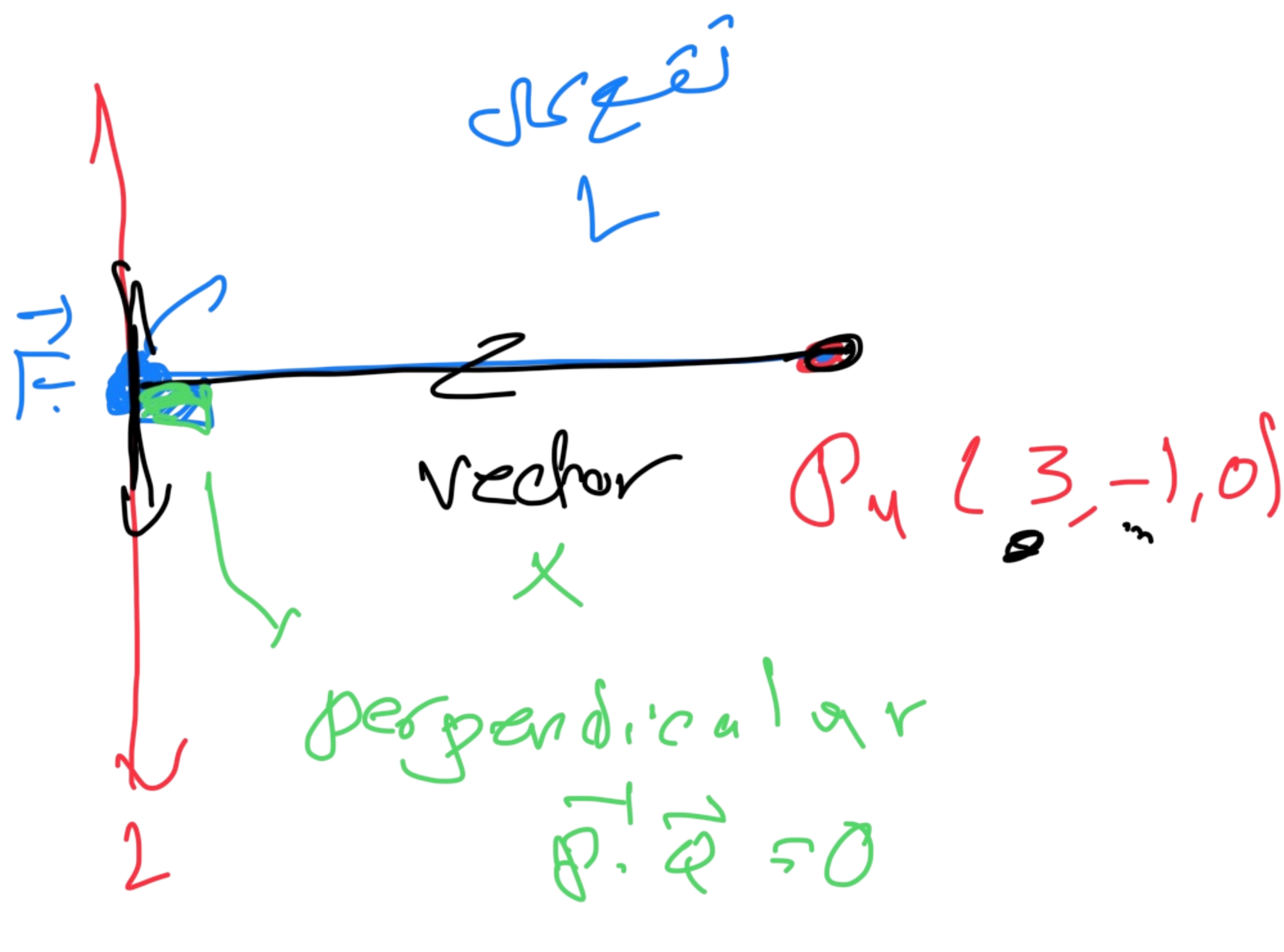
line equation :-

$$L = \text{point} + t \ast \text{vector parallel}$$

$$v = \langle -4, -1, 6 \rangle$$

$$\text{point} = P_2 = (1, 1, 2)$$

$$L = \langle 1, 1, 2 \rangle + t \langle -4, -1, 6 \rangle$$



$$F(\underbrace{1-4t}_{x},\ \underbrace{1-t}_{y},\ \underbrace{2+6t}_{z})$$

$$\vec{FP} = \vec{P} - \vec{F}$$

$$= \langle 2+4t,\ -2+t,\ -2-6t \rangle$$

$$\vec{\text{vector}}_2 \cdot \vec{\text{vector}}_x = 0$$

$$\langle -4,\ -1,\ 6 \rangle$$

$$\langle 2+4t,\ -2+t,\ -2-6t \rangle$$

$$-8-16t+2-t+-12-36t=0$$

$$-53t-18=0$$

$$\frac{\cancel{53}t}{\cancel{53}} = \frac{-18}{53}$$

$$\boxed{t = \frac{-18}{53}}$$

vector v

$$\left\langle \frac{34}{53}, \frac{-124}{53}, \frac{2}{53} \right\rangle$$

$$\text{distance} = \|\ \| = \sqrt{\left(\frac{34}{53}\right)^2 + \left(\frac{124}{53}\right)^2 + \left(\frac{2}{53}\right)^2}$$

$$\boxed{d = 2{,}4263}$$