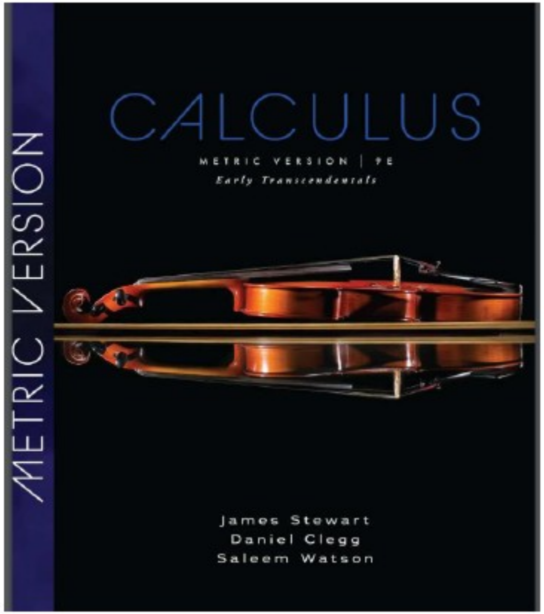
Calculus I



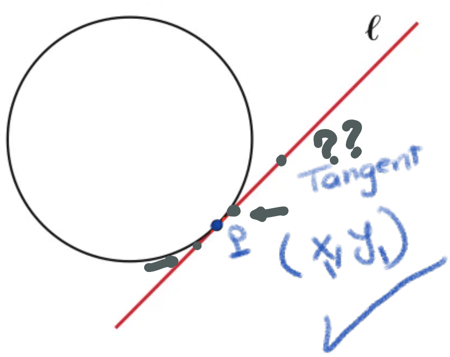
This helping matiral is taken from Stewart, J., Clegg, D.K., & Watson, S. (2021). Calculus: Early Transcendentals (9th ed.).

2.1 The Tangent and Velocity Problems

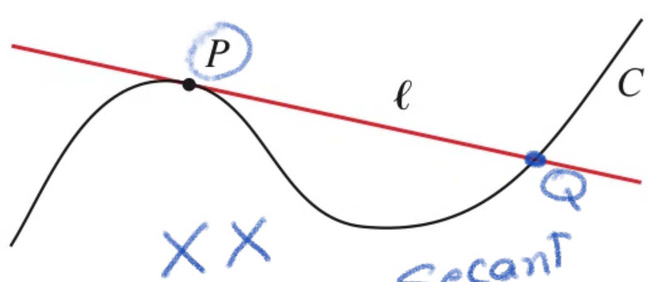
In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." We can think of a tangent to a curve as a line that touches the curve and follows the same direction as the curve at the point of contact. How can this idea be made precise?



(a)



(b)

Secant
قاطع

Equation of tangent line to the curve.

$\boxed{1}$ point

نقطة التماس $(x_1, y_1) = ✓$

$\boxed{2}$ Slope

m ✓

Math 1/1 ← $m = \frac{y_2 - y_1}{x_2 - x_1} = ✓$

Equation.

$$\boxed{y - y_1 = m(x - x_1)}$$

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	$\boxed{1.999}$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	$\boxed{2.001}$

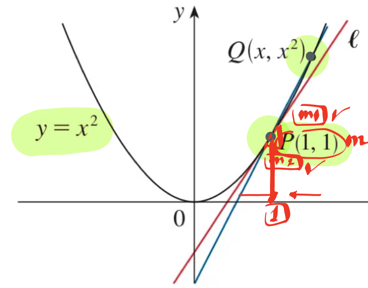
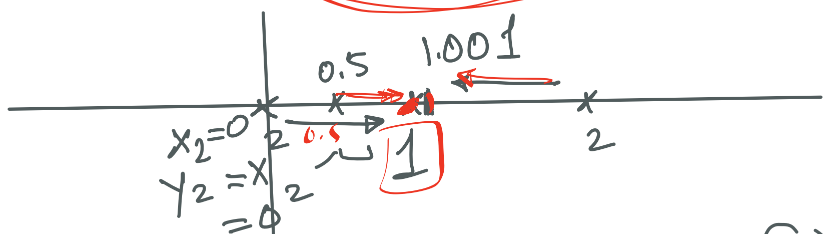


FIGURE 2

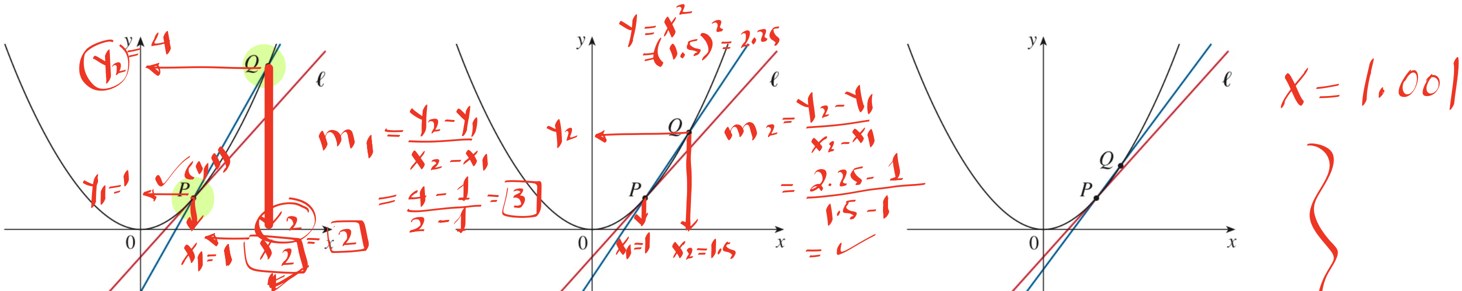
$x = 1 \Longrightarrow m = 2$



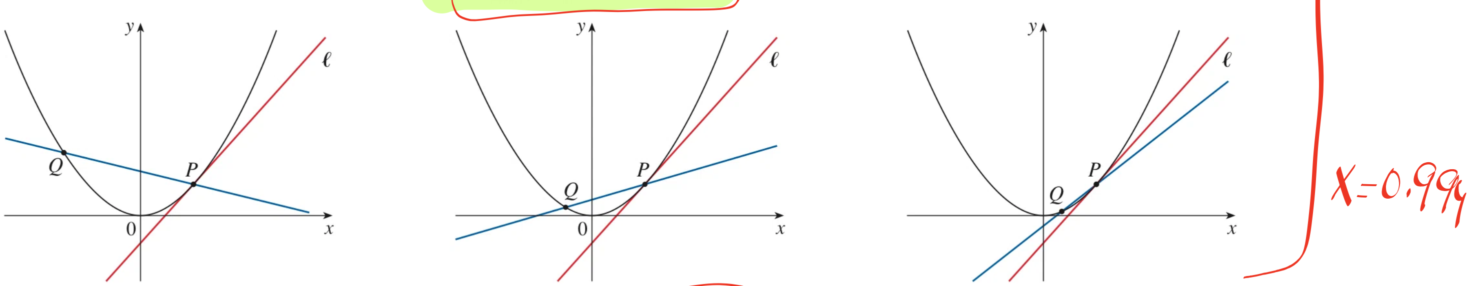
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 1}{0 - 1} = \frac{-1}{-1} = 1$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through $(1, 1)$ as

$y = x^2 = 2^2 = 4$



Q approaches P from the right



Q approaches P from the left

$$\text{Slope} = m = \frac{1.999 + 2.001}{2}$$

$$\boxed{m = 2}$$

ميل الخط
المستقيم
الذى يمس المنحنى
عند نقطة

$$y = x^2$$

$$(1, 1)$$

Equation

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 2 + 1$$

$$\boxed{y = 2x - 1}$$

$y = x^2$ at point $(1,1)$

9. The point $P(1, 0)$ lies on the curve $y_2 = \sin(10\pi/x)$.

(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2$, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8, and 0.9. Do the slopes appear to be approaching a limit?

(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.

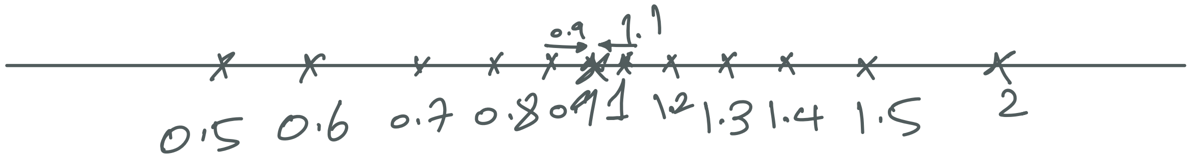
(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

x_2 $\left(x_2, \sin \frac{10\pi}{x}\right)$

x	$Q\,(x_2, y_2)$	$m_{PQ} = \frac{y_2}{x_2 - 1}$
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

ميل

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173



Slope

a)
As x approaches to 1, the slopes don't appear to be approaching any particular value

$$\text{slope} = m = \frac{Y_2 - Y_1}{X_2 - X_1}$$

$P\left(\overset{x_1}{1}, \overset{y_1}{0}\right)$

$$\boxed{m = \frac{Y_2 - 0}{X_2 - 1} = \frac{Y_2}{X_2 - 1}}$$

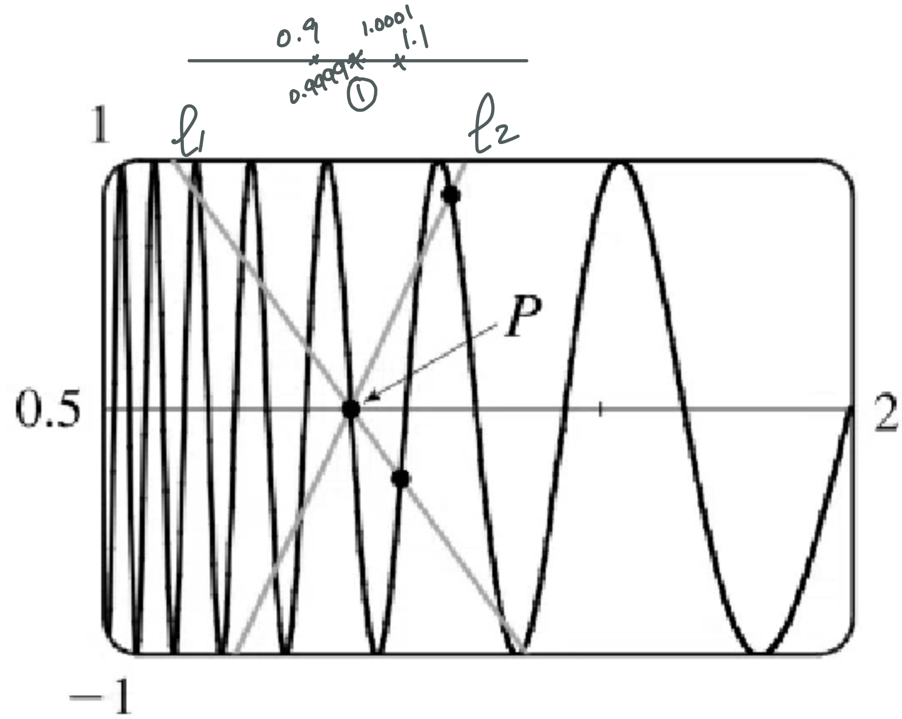
$$X_2 = 0.5 \longrightarrow Y_2 = \frac{\sin 10\pi}{X} = \frac{\sin 10\pi}{0.5} = 0$$

$$m = \frac{Y_2}{X_2 - 1} = \frac{0}{0.5 - 1} = 0$$

$$X_2 = 0.6 \longrightarrow Y_2 = \frac{\sin 10\pi}{X_2} = \frac{\sin 10\pi}{0.6} = 0.8660$$

$$m = \frac{Y_2}{X_2 - 1} = \frac{0.8660}{0.6 - 1} = -2.1651$$

(b)



c] By choosing a closer points to $x=1$.

$x=0.009$ $\quad$ $x=1$ $\quad$ $x=1.001$

$$Y=\sin\left(\frac{10\pi}{x}\right)$$

$$=\sin\left(\frac{10\pi}{0.009}\right)$$

$$Y=0.0314$$

from the left

$$m=\frac{Y_2-Y_1}{x_2-x_1}$$

$$=\frac{0.0314-0}{0.009-1}$$

$$=-31.4108$$

$$Y=\sin\left(\frac{10\pi}{x}\right)=\sin\left(\frac{10\pi}{1.001}\right)$$

$$Y=-0.0314$$

from the right

$$m=\frac{Y_2-Y_1}{x_2-x_1}$$

$$=\frac{-0.0314-0}{1.001-1}$$

$$=-31.3794$$

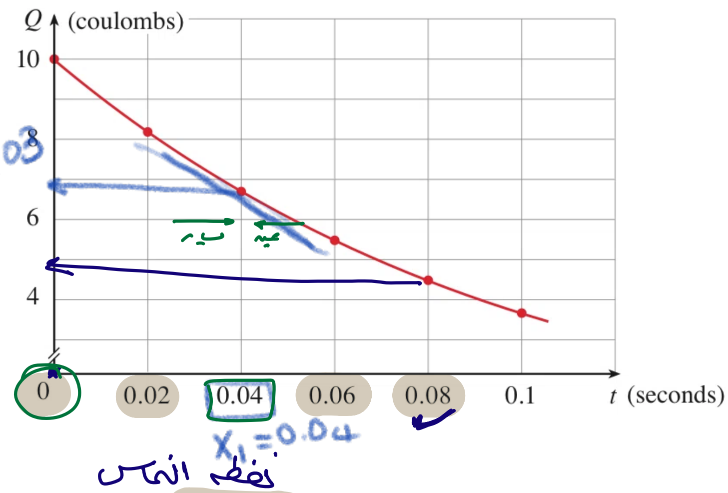
$$\text{Slope}=\frac{-31.4108+31.3794}{2}=\checkmark$$

زمن   الشحنه المتبقيه

t	Q
0	10
0.02	8.187
0.04	6.703
0.06	5.488
0.08	4.493
0.1	3.676

EXAMPLE 2 A pulse laser operates by storing charge on a capacitor and releasing it suddenly when the laser is fired. The data in the table describe the charge Q remaining on the capacitor (measured in coulombs) at time t (measured in seconds after the laser is fired). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. (*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the laser [measured in amperes].)

SOLUTION In Figure 4 we plot the given data and use these points to sketch a curve that approximates the graph of the function.



يسار   يمين

0.04

$X_2 = 0$
$Y_2 = 10$

$$m_2 = \frac{Y_2 - Y_1}{X_2 - X_1} = \frac{10 - 6.703}{0 - 0.04} = ✓$$

$X_2 = 0.08$
$Y_2 = 4.493$

$$m_2 = \frac{Y_2 - Y_1}{X_2 - X_1} = \frac{4.493 - 6.703}{0.08 - 0.04} = ✓$$

Given the points $P(0.04, 6.703)$ and $R(0, 10)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{10 - 6.703}{0 - 0.04} = -82.425$$

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -74.20 and -60.75. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-74.20 - 60.75) = -67.475$$

So, by this method, we estimate the slope of the tangent line to be about -67.5.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 5.

$$m = \frac{\overset{\text{يسار}}{m_2} + \overset{\text{يمين}}{m_2}}{2} = ✓$$

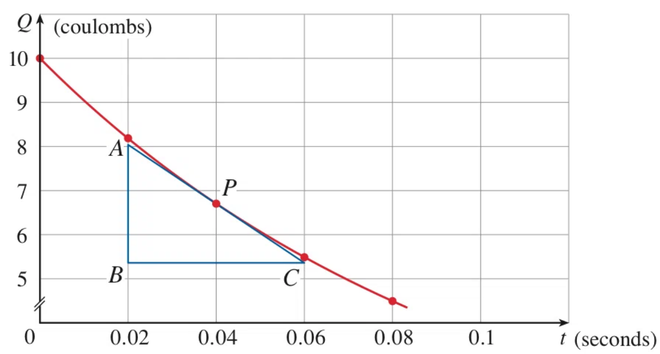


FIGURE 5

The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the laser after 0.04 seconds is about -65 amperes.

This gives an estimate of the slope of the tangent line as

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Summary

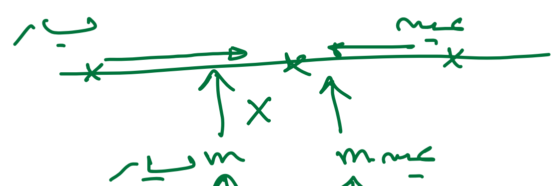
Tangent Line ملخص

Equation

Point
نقطة التماس

Slope
m
ميل الخط المستقيم

تخمين Estimate



متوسط الميول average

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

نقطة التماس

نقاط يمين او يسار

Equation $y - y_1 = m(x - x_1)$

ميل المماس