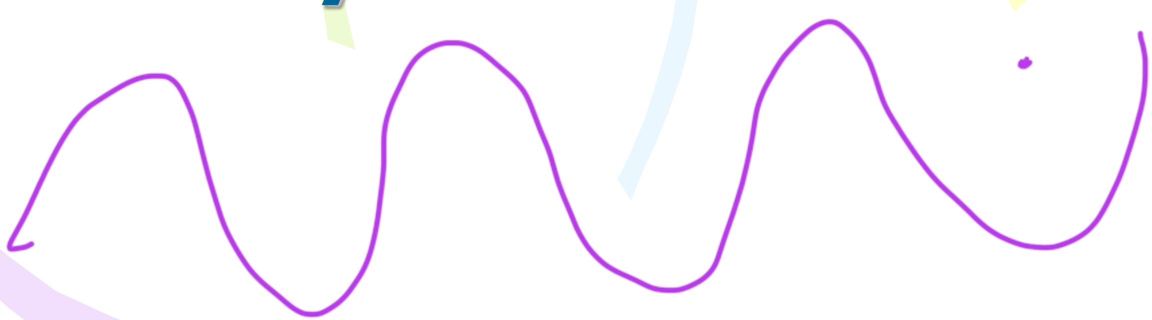
AC Circuits:
Sinusoidal Steady-State Analysis



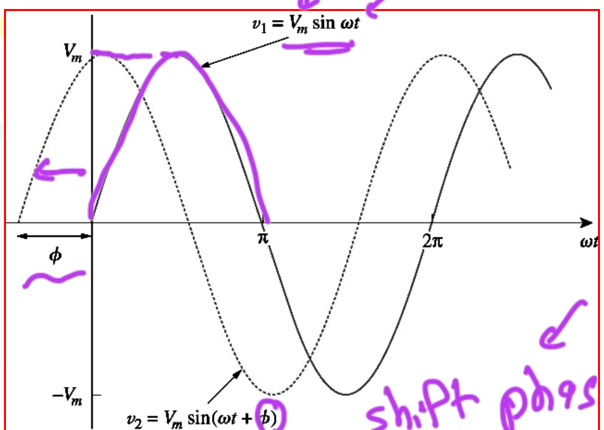
Sinusoids and Phasor

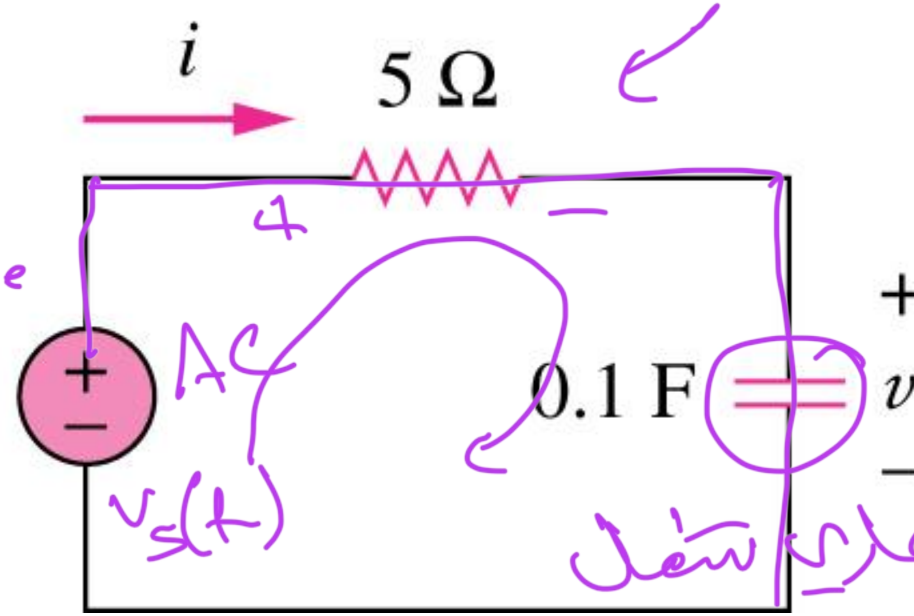
- Motivation
- Sinusoids’ features
- Phasors
- Phasor relationships for circuit elements
- Impedance and admittance
- Kirchhoff’s laws in the frequency domain
- Impedance combinations

Motivation (1)

DC → C → off → open

How to determine *v(t)* and *i(t)*?





$v_s(t) = 10V$

$I_c = C \frac{dv}{dt}$

How can we apply what we have learned before to determine *i(t)* and *v(t)*?

$-V_s + 5i' + V = 0$

AC

$5i' + v(t) = V_s$

$$\boxed{5 \times 0.1 \frac{dv(t)}{dt} + v(t) = V_s}$$

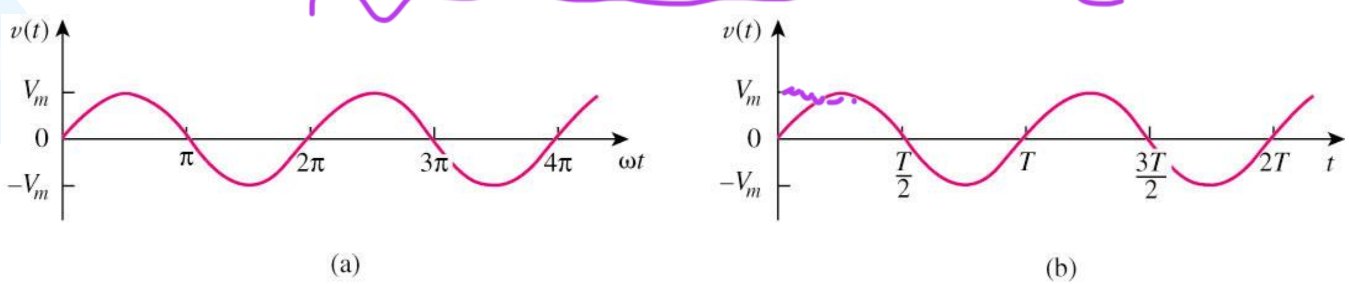
First order diff. eq

Sinusoids (1)

AC

- A sinusoid is a signal that has the form of the sine or cosine function.
- A general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi)$$



A sketch of $V_m \sin \omega t$: (a) as a function of ωt, (b) as a function of t.

where

Vm = the **amplitude** of the sinusoid

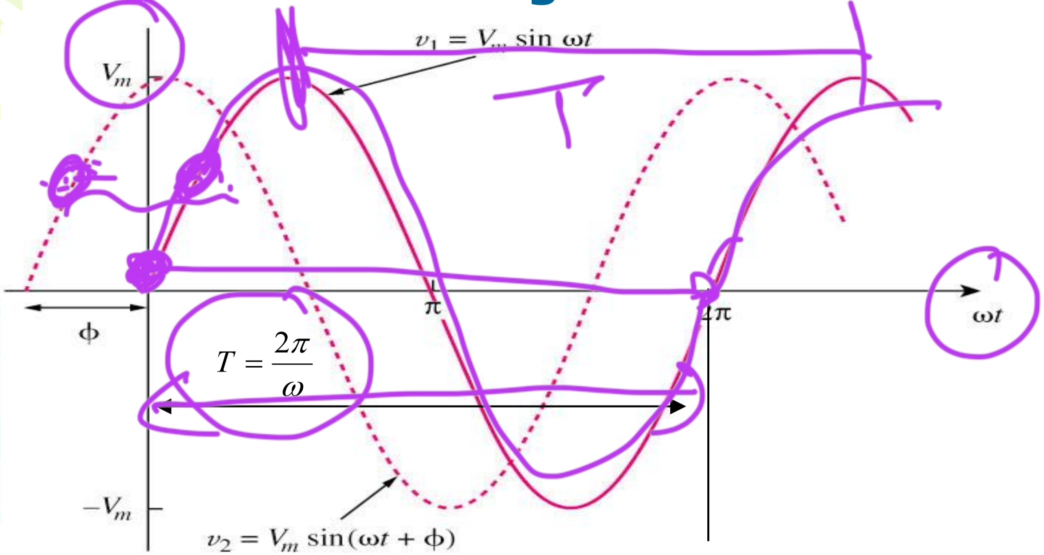
ω = the angular frequency in radians/s rad/s

Φ = the phase (Φ =0 in above sketch)

ωt= the *argument* of the sinusoid

Sinusoids (2)

A periodic function is one that satisfies *v(t) = v(t + nT)*, for all t and for all integers n.



$$f = \frac{1}{T} Hz$$

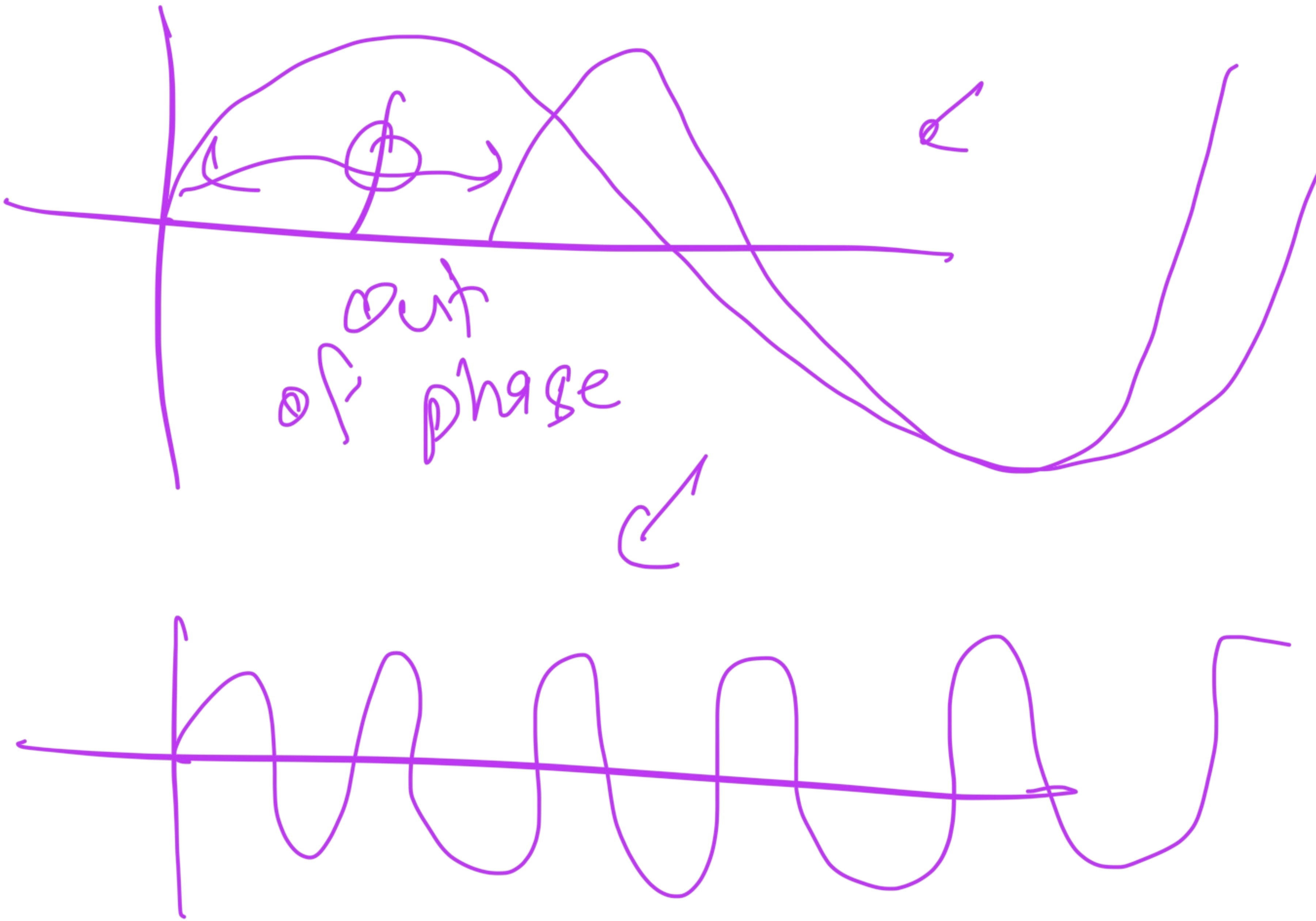
$$\omega = 2\pi f$$

$T = \frac{1}{f}$ $f = \frac{1}{T}$

The starting point of v_2 in Fig. occurs first in time.

Therefore, we say that v_2 *leads* v_1 by ϕ or that v_1 *lags* v_2 by ϕ. If $\phi \neq 0$, we also say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*;

- Only two sinusoidal values with the same frequency can be compared by their amplitude and phase difference.
- If phase difference is zero, they are in phase; if phase difference is not zero, they are out of phase.


ϕ
out of phase

We can transform a sinusoid from sine form to cosine form or vice versa.

$$\sin(\omega t \pm 180^\circ) = -\sin\omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos\omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos\omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin\omega t$$

$-\sin(\omega t) \rightarrow +\sin(\omega t \mp 180)$

$-\cos(\omega t) \rightarrow +\cos(\omega t \mp 180)$

$\pm\sin(\omega t) = \cos(\omega t \mp 90)$

$\mp\sin(\omega t) = \cos(\omega t \pm 90)$

$\pm\cos(\omega t) = \sin(\omega t \pm 90)$

$\mp\cos(\omega t) = \sin(\omega t \mp 90)$

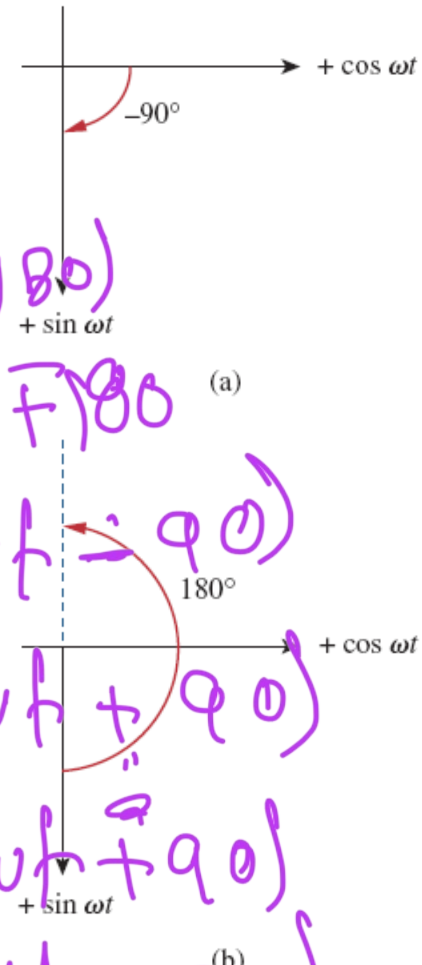


Figure 9.3
A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90^\circ) = \sin\omega t$, (b) $\sin(\omega t + 180^\circ) = -\sin\omega t$.

Sinusoids (3)

$\frac{4\pi}{2\pi} = \omega = 2\pi f$

$f = 2\text{ Hz}$

Example 1

Given a sinusoid, $5\sin(4\pi t - 60^o)$, calculate its amplitude, phase, angular frequency, period, and frequency.

2Hz   5 -60° ω 4π rad/s $T = \frac{1}{f} = \frac{1}{2} = 0.5$ s

time

Solution:

Amplitude = 5, phase = –60^o, angular frequency = 4π rad/s, Period = 0.5 s, frequency = 2 Hz.

$$\omega = \frac{\cancel{2\pi} f}{\cancel{2\pi}} = \frac{50}{2\pi}$$

$$f = \frac{50}{2\pi}$$

Example 2

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

12   10 $\omega = 50$ $\frac{50}{2\pi}$ Hz

Solution:

$T = \frac{1}{f} = \frac{2\pi}{50}$ s

The amplitude is $V_m = 12$ V.

The phase is $\phi = 10^\circ$.

The angular frequency is $\omega = 50$ rad/s.

The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.

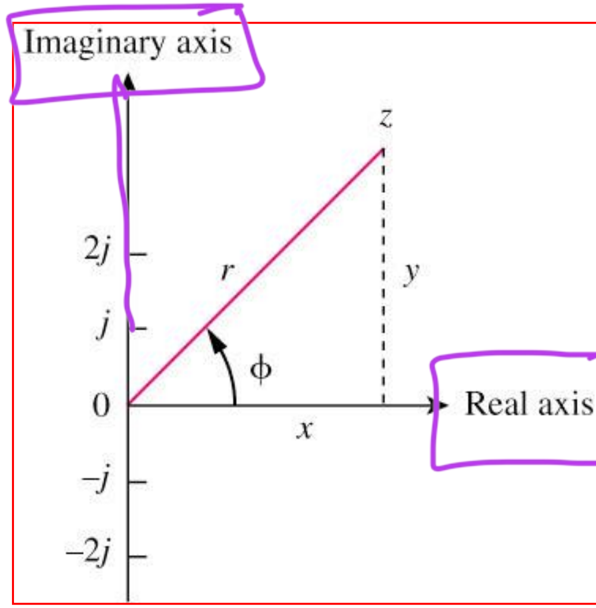
The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

complex number الأعداد المركبة

الآلة حاسبة

Phasor (1)

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.



- It can be represented in one of the following three forms:

a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

b. Polar $z = r\angle\phi$

c. Exponential $z = re^{j\phi}$

where
$$r = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$

الأعداد العادية π , 6 , -5

, e , $\sqrt{3}$, ----

الأعداد المركبة

$i = \sqrt{-1}$

$\boxed{5 + 3j}$

real part 5

Imag part 3

$-5 - 7j$

$+10j \leftarrow 5i$

1. Rectangular $\quad 5 + 3i$

2. $\boxed{\text{Polar}}$ $\longrightarrow$ $10 \angle 30$

3. Expon $\longrightarrow$ $10\, e^{j30}$

Phasor (3)

Mathematic operation of complex number:

1. Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. Multiplication $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$
4. Division $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \angle \phi_1 - \phi_2$
5. Reciprocal $\dfrac{1}{z} = \dfrac{1}{r} \angle -\phi$
6. Square root $\sqrt{z} = \sqrt{r} \angle \phi/2$
7. Complex conjugate $z^* = x - jy = r \angle -\phi = re^{-j\phi}$
8. Euler's identity $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

$\angle \Rightarrow \cos(\omega t$

Phasor (4)

- Transform a sinusoid to and from the time domain to the phasor domain:

$$v(t) = V_m \cancel{\cos(\omega t} + \phi) \longleftrightarrow V = V_m \angle \phi$$

(time domain) (phasor domain)

- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the cosine function in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting from the phase.

Phasor (2)

Example 2

Try it at home.

- Evaluate the following complex numbers:

a. $$[(5 + j2)(-1 + j4) - 5\angle 60^\circ]$$

b. $$\frac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ$$

Solution:

a. −15.5 + j13.67

b. 8.293 + j2.2

$$\left[\underbrace{(5+j2)(-1+j4)}\div\underbrace{(5\angle 60)}\right] =$$

$$\frac{-15.5+j13.6699}{\left(\frac{(10+j5)+(3\angle 40)}{(-3+j4)}\right)+(10\angle 30)}$$

$$8.293+j2.2$$

Phasor (5)

Example 3

Transform the following sinusoids to phasors:

$i(t) = 6\cos(50t - 40^o)$ A

$v(t) = -4\sin(30t + 50^o)$ V

لازم تكون cos

Solution:

a. $I = 6\angle -40^\circ$ A

b. Since $-\sin(A) = \cos(A+90^o)$;

$v(t) = 4\cos(30t+50^o+90^o) = 4\cos(30t+140^o)$ V

Transform to phasor => $V = 4\angle 140^\circ$ V

$$i(t) = 6\,\cancel{\cos(\omega t} - 40)\ A$$

$$\boxed{I = 6\angle -40\ A}$$

$$v(t) = -4\sin(30t + 50)$$

$$-\sin(\omega t) \longrightarrow \cos(\omega t + 90)$$

$$v(t) = +4\,\cancel{\cos(30t} + \underbrace{50 + 90}_{140})$$

$$\boxed{V = 4\angle 140\ V}$$

Phasor (6)

Example 4:

Transform the sinusoids corresponding to phasors:

a. $\mathbf{V} = -10\angle 30^\circ \text{ V}$

b. $\mathbf{I} = j(5 - j12) \text{ A}$

Solution:-10 /30 = (-1)(10/30)= (-1+j0)(10/30)= (1/180)(10/30)=10/210

a) *v(t) = 10cos(ωt + 210°)* V

b) Since $\mathbf{I} = 12 + j5 = \sqrt{12^2 + 5^2} \angle \tan^{-1}(\frac{5}{12}) = 13\angle 22.62^\circ$

i(t) = 13cos(ωt + 22.62°) A

$r\angle\theta \longrightarrow -r\angle\theta+\pi=180$

$10\angle 30 \longrightarrow$

$10\angle 30+180 \longrightarrow \boxed{10\angle 210 \quad \cos(\omega t+}$

$v(t) = 10 \ \underline{\cos(\omega t+210)}$

$I = \underline{(j(25-j12)} \longrightarrow 12+j25$

shift → 2 → 3

$13\angle 22.62$

$i(t) = 13\cos(\omega t+22.62)$ A

Phasor (7)

The differences between v(t) and V:

- v(t) is instantaneous or time-domain representation
  V is the frequency or phasor-domain representation.
- v(t) is time dependent, V is not.
- v(t) is always real with no complex term, V is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the same frequency.

Phasor diagram

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A phasor diagram showing $\mathbf{V} = V_m \angle \phi$ and $\mathbf{I} = I_m \angle -\theta$.

V lead I

I lag V

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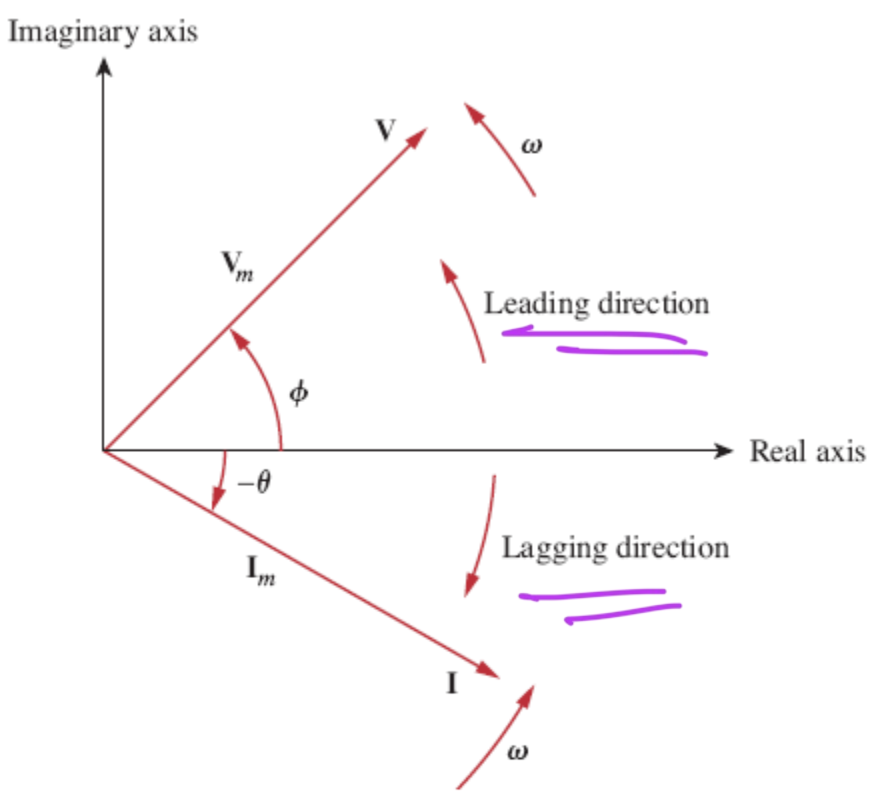


Sinusoid-phasor transformation

Time domain representation		Phasor domain representation
$V_m \cos(\omega t + \phi)$		$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$\rightarrow \cos(\omega t - 90)$	$V_m \angle \phi - 90°$
$I_m \cos(\omega t + \theta)$		$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$		$I_m \angle \theta - 90°$

Phasor (8)

Relationship between differential, integral operation in phasor listed as follow:

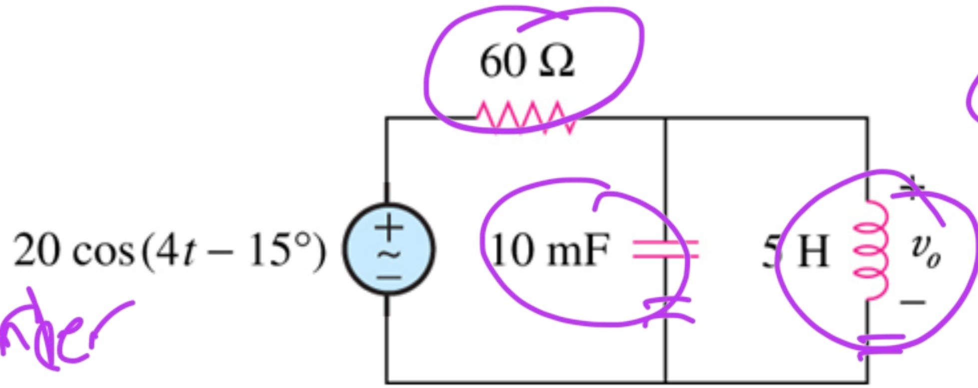
$$v(t) \longleftrightarrow \mathbf{V} = V\angle\phi$$

$$\frac{dv}{dt} \longleftrightarrow j\omega \mathbf{V}$$

$$\int v\,dt \longleftrightarrow \frac{\mathbf{V}}{j\omega}$$

Phasor (10)

- we can derive the differential equations for the following circuit in order to solve for $v_o(t)$ in phase domain V_o.



2nd order diff eq

$$\frac{d^2 v_o}{dt^2} + \frac{5}{3}\frac{dv_0}{dt} + 20v_0 = -\frac{400}{3}\sin(4t - 15^o)$$

- However, the derivation may sometimes be very tedious.

Is there any quicker and more systematic methods to do it?

Phasor (11)

The answer is YES!

Instead of first deriving the differential equation and then transforming it into phasor to solve for V_o, we can transform all the RLC components into phasor first, then apply the KCL laws and other theorems to set up a phasor equation involving V_o directly.

time domain

Phasor

$V_m \angle \phi$

KCL

KVL

node

mesh

Phasor Relationships for Circuit Elements (2)

Summary of voltage-current relationship		
Element	Time domain	Phasor domain
R	$v = Ri$	$V = RI$, Z
L	$v = L\dfrac{di}{dt}$	$V = j\omega L I$ Z
C	$i = C\dfrac{dv}{dt}$	$V = \dfrac{I}{j\omega C}$ Z

$\boxed{R} = \dfrac{1}{\sigma}$

$\sigma = \dfrac{1}{R}$

$R \Leftarrow \boxed{Z} = \frac{1}{Y}$ $\qquad$ $Y = \frac{1}{Z}$

Impedance and Admittance (1)

$C, L \Rightarrow R$

- The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms Ω.

From Ohm's Law: $$Z = \frac{V}{I} = R + jX$$

complex $\quad Z = 5 + j3$

Real $\qquad$ imag

where R = Re [Z] is the resistance and X = Im [Z] is the reactance. Positive X is for L and negative X is for C.

- The admittance Y is the reciprocal of impedance, measured in siemens (S).

$$Y = \frac{1}{Z} = \frac{I}{V}$$

$Z = 5 + j8 \rightarrow L$

$Z = 1 - j2 \rightarrow C$

phasor → KVL KCL
node mesh

Impedance and Admittance (2)

Impedances and admittances of passive elements

Element	Impedance	Admittance
R	$Z = R$	$Y = \dfrac{1}{R}$
L	$Z = j\omega L$	$Y = \dfrac{1}{j\omega L}$
C	$Z = \dfrac{1}{j\omega C}$	$Y = j\omega C$

time $\Longrightarrow$ Freq

phasor

Impedance and Admittance (3)

After we know how to convert RLC components from time to phasor domain, we can transform a time domain circuit into a phasor/frequency domain circuit.

Hence, we can apply the KCL/KVL laws and other theorems to directly set up phasor equations involving our target variable(s) for solving.

Phasor Relationships for Circuit Elements (4)

Example 6

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

time

Solution: $\omega = 60$ rad/s

$v = 12\cos(60t + 45°)$, and $\mathbf{V} = 12\underline{/45°}$ V.

For the inductor, $Z = j\omega L = j60 \times 0.1 = 6\underline{/90°}\ \Omega$

Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45°}}{6\underline{/90°}} = 2\underline{/-45°}\ \text{A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°)\ \text{A}$$

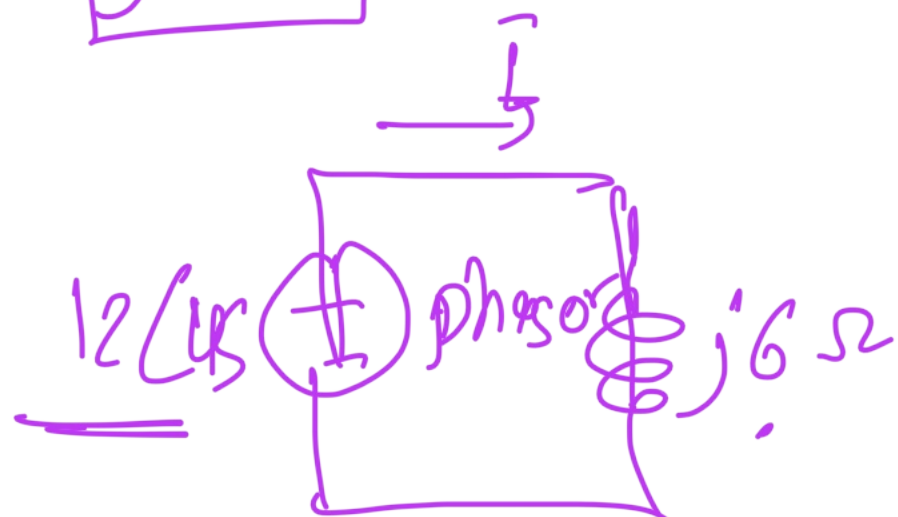
$V = 12\cos(60t + 45)$ $L = 0.1\,H$

$\omega = 60$

$Z_L = j\omega L$

$Z_L = j * 60 * 0.1 = \boxed{j6\,\Omega}$

$V = 12\angle 45$



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$\boxed{V = IZ}$ $V = IR$

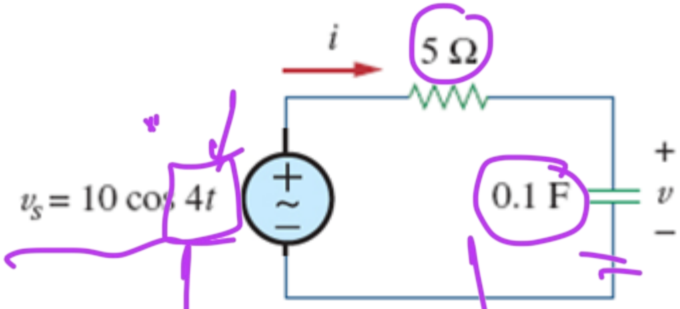
$$\Rightarrow I = \frac{V}{Z} = \frac{(12\angle 45)}{(j6)} = 2\angle -45$$

$$\boxed{i(t) = 2\cos(60t - 45)}\ A$$

$$\boxed{V_L = L\frac{di}{dt}} \rightarrow \boxed{i = \frac{1}{L}\int v(t)}$$

Impedance and Admittance (5)

Example 7 Find $v(t)$ and $i(t)$ in the circuit shown



$\omega = 4$

time domain

Solution:

From the voltage source $10 \cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

The voltage across the capacitor is

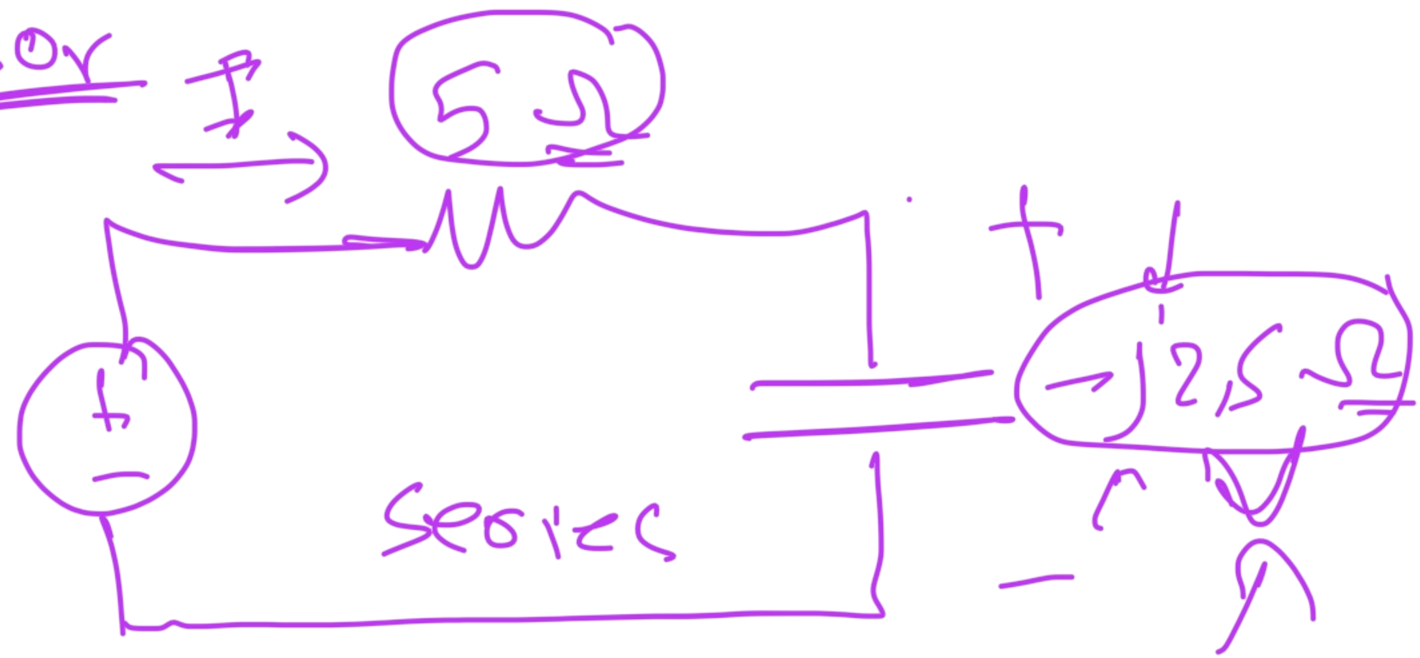
$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$
$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V}$$

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

$$\omega = 4 \qquad Z_C = \frac{1}{j\omega C} = \frac{1}{j \times 4 \times 0.1} = -j2.5\ \Omega$$

Phasor

$10\angle 0$



$$I = \frac{V}{\Sigma Z} = \frac{(10\angle 0)}{(5 - j2.5)}$$

$$\boxed{I = 1{,}789 \angle 26{,}57 \text{ A}}$$

$$V = I Z_C = (1{,}789 \angle +26{,}57)(-j7{,}5)$$

$$\boxed{V = 4{,}47 \angle -63{,}43}$$

$$\boxed{i(t) = 1{,}789 \cos(4t + 26{,}57) \text{ A}}$$

$$\boxed{v(t) = 4{,}47 \cos(4t - 63{,}43) \text{ V}}$$

Summary:

- Circuits analysis are hold in the phasor domain or more commonly called frequency domain.
- Moreover, the variables to be handled (voltage or current) are phasors, which are complex numbers.
- All the mathematical operations involved are now in complex domain.

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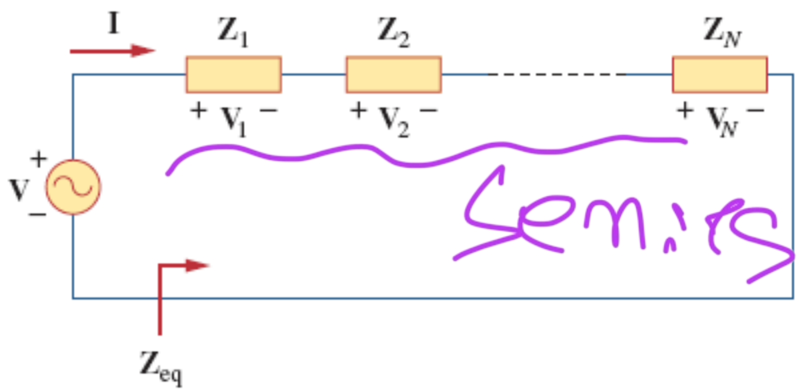
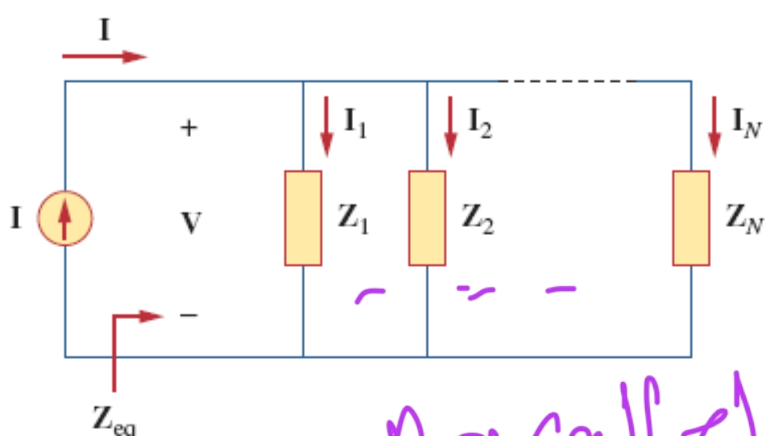
Impedance Combinations (1)



series

$$\boxed{Z_{eq} = Z_1 + Z_2 + \cdots + Z_N}$$



parallel

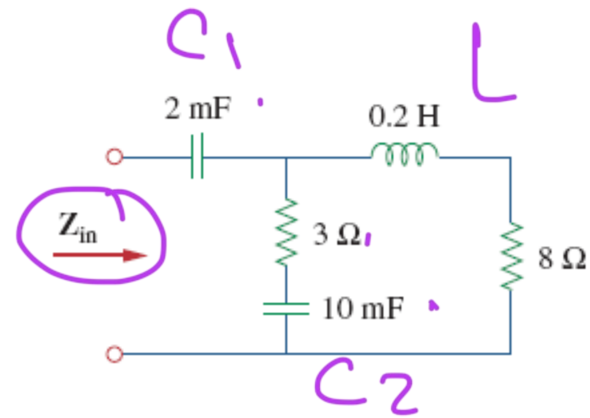
$$\frac{1}{Z_{eq}} = \frac{I}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}$$

$$\boxed{Y_{eq} = Y_1 + Y_2 + \cdots + Y_N}$$

Impedance Combinations (2)

Example 8

Find the input impedance of the circuit. Assume that the circuit operates at $\omega = 50$ rad/s.



Solution:

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

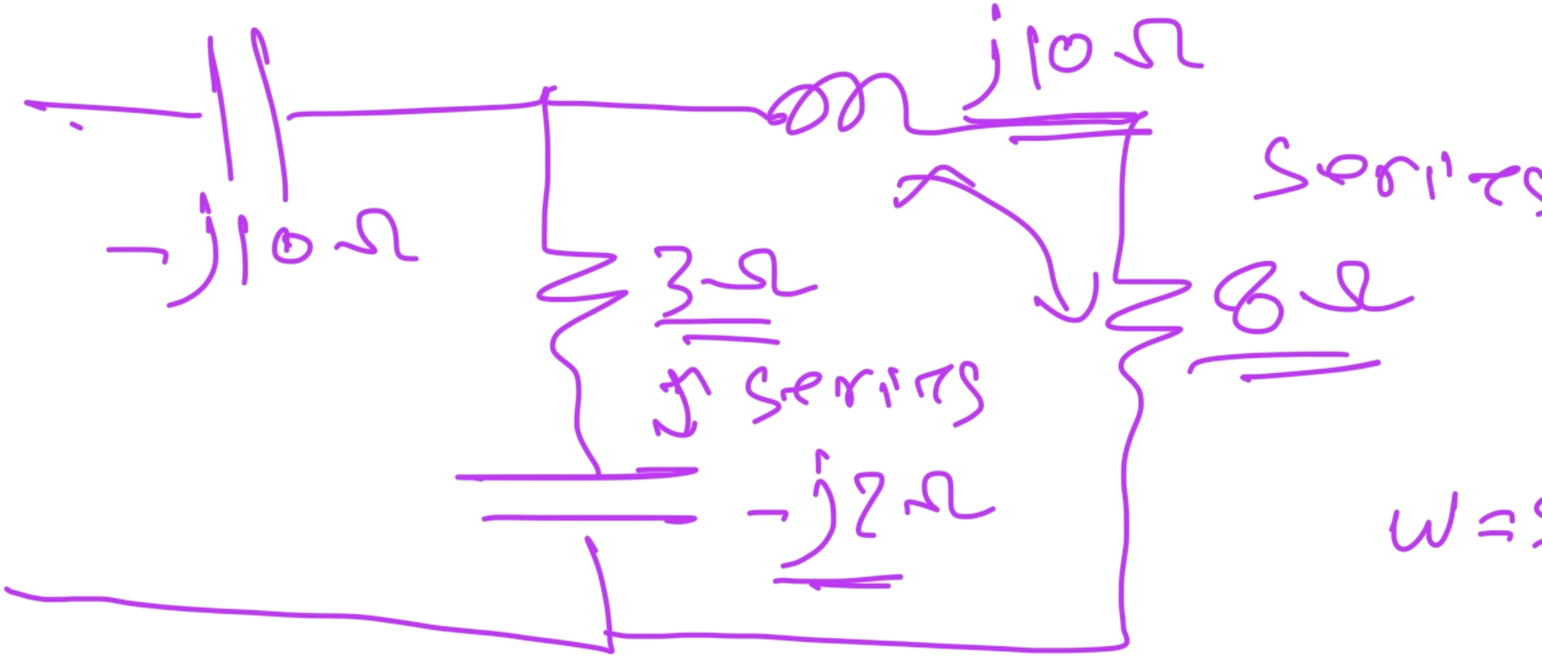
$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07 = 3.22 - j11.07\ \Omega$$

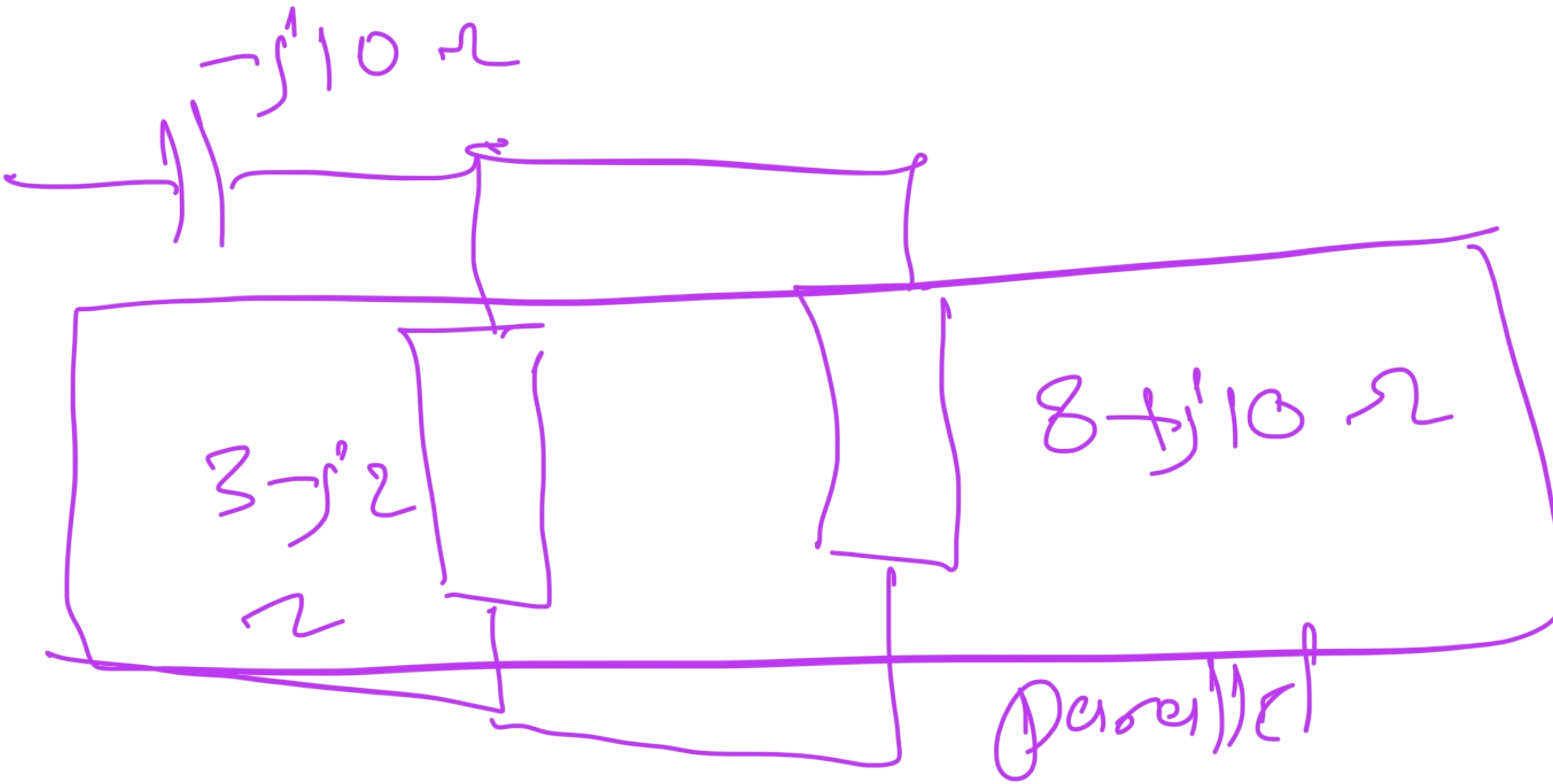


$\omega = 50$

$$Z_{C1} = \frac{1}{j\omega C_1} = \frac{1}{50 \times 2 \times 10^{-3} \times j} = -j10\,\Omega$$

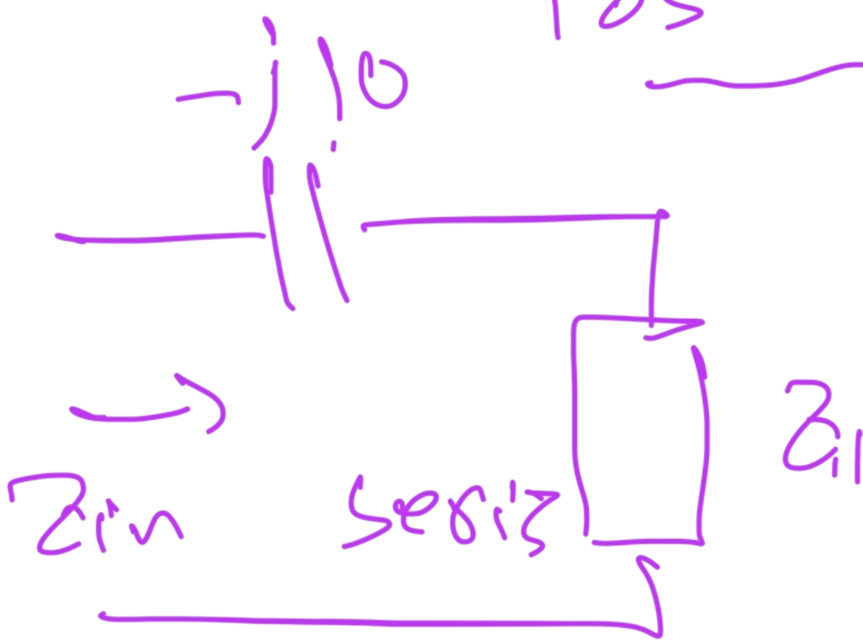
$$Z_{C2} = \frac{1}{j\omega C_2} = \frac{1}{50 \times 10 \times 10^{-3}\, j} = -j2\,\Omega$$

$$Z_L = j\omega L = 50 \times 0{,}2\, j = j10$$



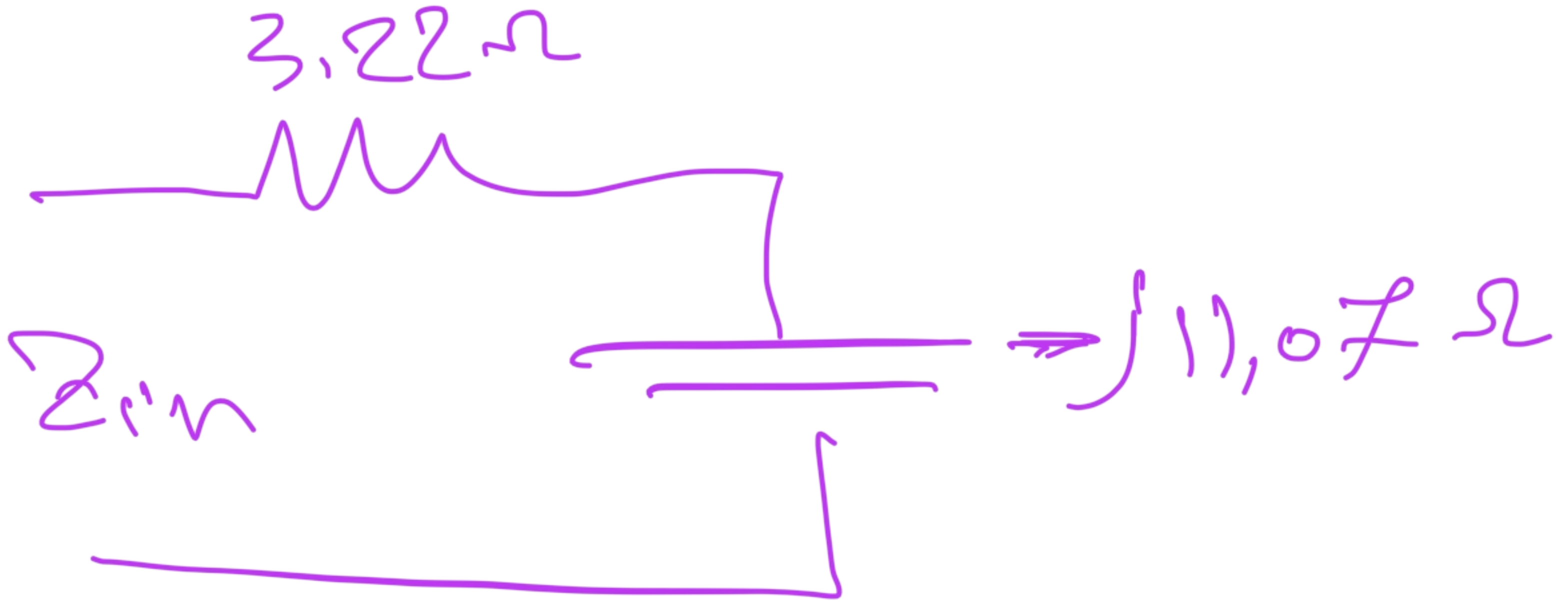
$$Z_1 = \frac{1}{\frac{1}{(3-j2)} + \frac{1}{(8+j10)}}$$

$$Z_1 = \frac{596}{185} - j\,\frac{198}{185}$$



$$Z_{in} = \frac{596}{185} - j\,\frac{198}{185} - j10 \quad C$$

$$Z_{in} = 3{,}22 - j\,11{,}07\ \Omega$$



3,22 Ω
Zin
⇒ j11,07 Ω