الهيئة الملكية للجبيل وينبع
Royal Commission for Jubail & Yanbu

كلية الجبيل الصناعية
JUBAIL INDUSTRIAL COLLEGE

EEN314: Signals and Systems

Lecture 11

Lecture Outline

- The Fourier Transform 5.2: Properties of Fourier Transform

The Fourier Transform 5.2: Properties of Fourier Transform

TABLE 5.1 Fourier Transform Properties

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time reversal	$f(-t)$	$F(-\omega)$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
Modulation (Multiplication)	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$
Differentiation in time	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
Differentiation in Frequency	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Symmetry	$f(t)$ real	$F(-\omega) = F^*(\omega)$

The Fourier Transform 5.2: Properties of Fourier Transform

- **Linearity:**

$$f_1(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(\omega) \quad \text{and} \quad f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_2(\omega) \Rightarrow [af_1(t) + bf_2(t)] \overset{\mathcal{F}}{\longleftrightarrow} [aF_1(\omega) + bF_2(\omega)]$$

EXAMPLE 5.3 **The linearity property of the Fourier transform**

We can make use of the property of linearity to find the Fourier transforms of some types of waveforms. For example, consider

$$f(t) = B \cos \omega_0 t.$$

$$f(t) = \frac{B}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}] = \frac{B}{2}e^{j\omega_0 t} + \frac{B}{2}e^{-j\omega_0 t}.$$

$$\mathcal{F}\{B \cos \omega_0 t\} = \frac{B}{2}F\{e^{j\omega_0 t}\} + \frac{B}{2}F\{e^{-j\omega_0 t}\}$$

$$= \pi B\delta(\omega - \omega_0) + \pi B\delta(\omega + \omega_0)$$ Let's plot it

$$B\cos \omega_0 t \overset{\mathcal{F}}{\longleftrightarrow} \pi B[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

The Fourier Transform 5.2: Properties of Fourier Transform

- **Time Shifting:**

$$f(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)e^{-j\omega t_0};$$

EXAMPLE 5.5 **The time-shifting property of the Fourier transform**

We now find the Fourier transform of the impulse function, which occurs at time zero. From (5.8),

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega t}\Big|_{t=0} = 1.$$

If the impulse function is shifted in time so that it occurs at time t_0 instead of at $t = 0$, we see

$$\mathcal{F}\{\delta(t - t_0)\} = (1)e^{-j\omega t_0} = e^{-j\omega t_0},$$

The Fourier Transform 5.2: Properties of Fourier Transform

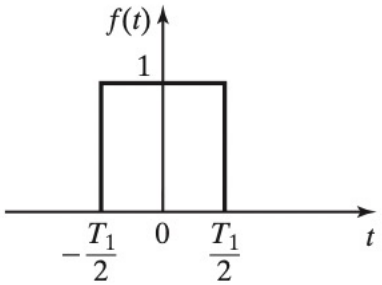
- **Time Scaling:**

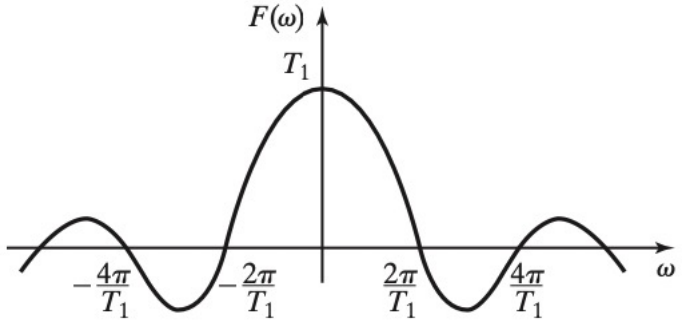
$$f(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

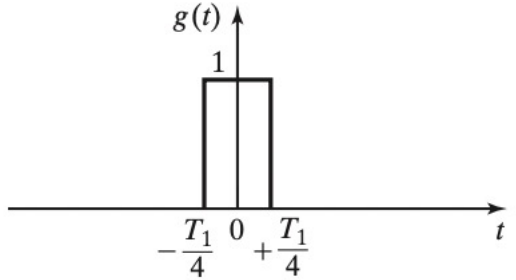
EXAMPLE 5.4 **The time-scaling property of the Fourier transform**

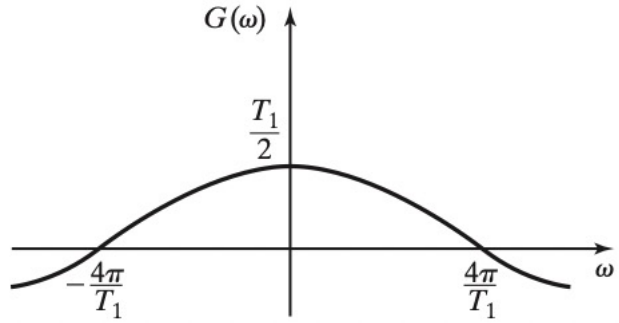
We now find the Fourier transform of the rectangular waveform

$$g(t) = \text{rect}(2t/T_1).$$









$$G(\omega) = \frac{1}{2} F(\omega/2) = \frac{T_1}{2} \text{sinc}\left(\frac{\omega T_1}{4}\right)$$

The Fourier Transform 5.2: Properties of Fourier Transform

- **Time Reversal:**

$$f(-t) = \frac{1}{|-1|} F\left(\frac{\omega}{-1}\right) = F(-\omega).$$

EXAMPLE 5.7 **Fourier transform of a time-reversed function**

Consider the time-shifted impulse function,

$$g(t) = \delta(-t - t_0).$$

$$G(\omega) = e^{j\omega t_0}$$

The Fourier Transform 5.2: Properties of Fourier Transform

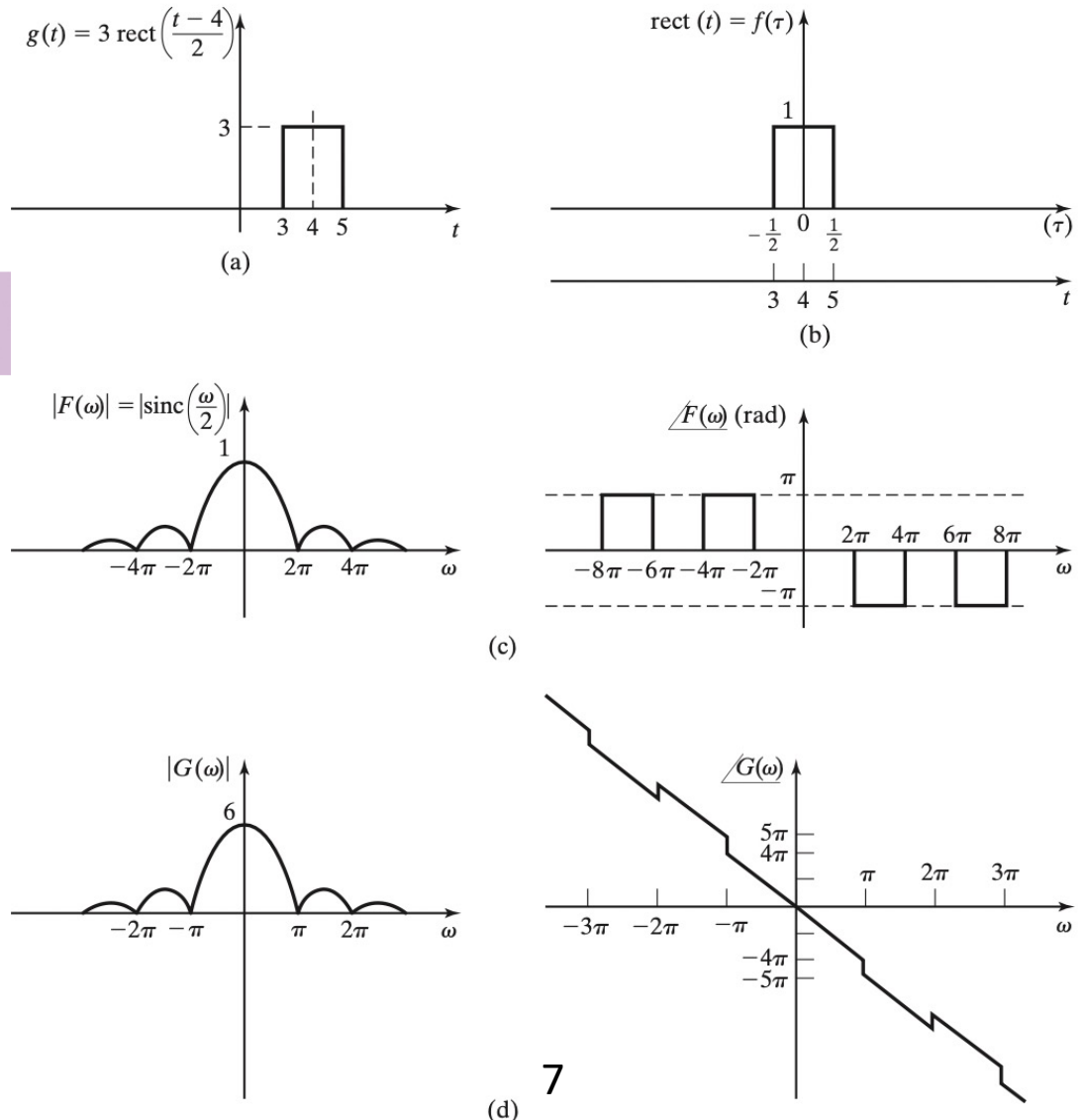
- **Time Transformation:**

$$\tau = at - t_0 \quad \Rightarrow \quad f(at - t_0) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-jt_0(\omega/a)}$$

EXAMPLE 5.8 **The time-transformation property of the Fourier transform**

$$g(t) = 3\,\text{rect}\,[(t-4)/2] = 3f(0.5t - 2)$$

$$G(\omega) = 6\,\text{sinc}(\omega)e^{-j4\omega}$$



The Fourier Transform 5.2: Properties of Fourier Transform

- **Duality:**

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt;$$

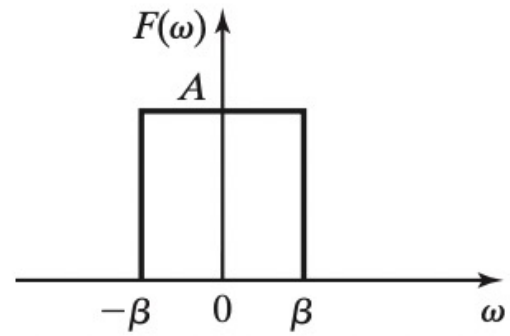
$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$F(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi f(-\omega) \quad \text{when} \quad f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)$$

EXAMPLE 5.9 **The duality property of the Fourier transform**

We now find the inverse Fourier transform of the waveform shown in Figure 5.8(a), using the transform pair derived in Example 5.1. Figure 5.8(a) shows a rectangular waveform, but in the frequency domain rather than in the time domain.

$$F(\omega) = A\,\text{rect}(\omega/2\beta).$$



- We know from a previous example:

$$V\,\text{rect}(t/T) \overset{\mathcal{F}}{\longleftrightarrow} TV\,\text{sinc}(\omega T/2)$$

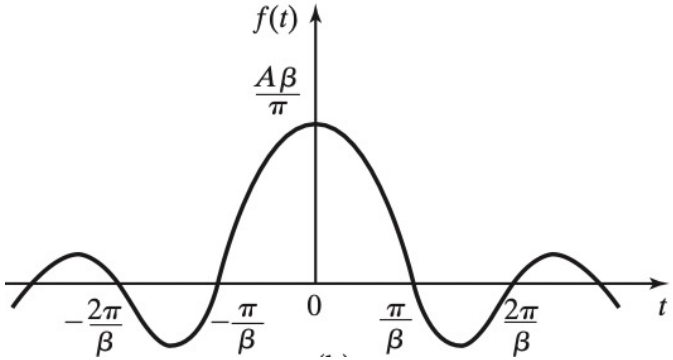
- So,

$$TV\,\text{sinc}(Tt/2) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi V\,\text{rect}(-\omega/T)$$

$$2\pi f(-\omega) = 2\pi A\,\text{rect}(\omega/2\beta)$$

$$F(t) = 2\beta A\,\text{sinc}(\beta t).$$

$$\frac{A\beta}{\pi}\,\text{sinc}\,(\beta t) \overset{\mathcal{F}}{\longleftrightarrow} \text{A rect}(\omega/2\beta)$$



The Fourier Transform 5.2: Properties of Fourier Transform

- **Convolution:**

$$f_1(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(\omega) \quad \text{and} \quad f_2 \overset{\mathcal{F}}{\longleftrightarrow} F_2(\omega)$$

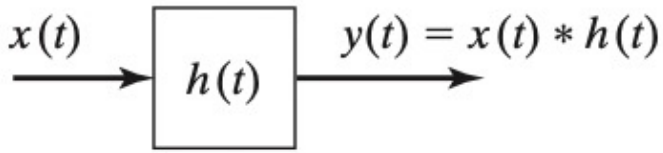
$$f_1(t)*f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(\omega)F_2(\omega), \qquad f_1(t)f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} F_1(\omega)*F_2(\omega)$$

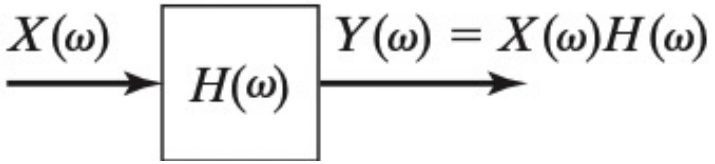
EXAMPLE 5.10 **The time-convolution property of the Fourier transform**

Chapter 3 discusses the response of linear time-invariant systems to input signals. A block diagram of a linear system is shown in Figure 5.9(a). If the output of the system in response to an impulse function at the input is described as $h(t)$, then $h(t)$ is called the *impulse response* of the system. The output of the system in response to any input signal can then be determined by convolution of the impulse response, $h(t)$, and the input signal, $x(t)$:

$$y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \qquad Y(\omega) = X(\omega)H(\omega)$$

$H(\omega)$ is transfer function





$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega), x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega), \quad \text{and} \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega)$$

The Fourier Transform 5.2: Properties of Fourier Transform

- **Frequency Shifting:** $x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0)$

EXAMPLE 5.12 **The frequency-shift property of the Fourier transform**

In the generation of communication signals, often two signals such as

$$g_1(t) = 2\cos(200\pi t) \quad \text{and} \quad g_2(t) = 5\cos(1000\pi t)$$

are multiplied together to give

$$g_3(t) = g_1(t)g_2(t) = 10\cos(200\pi t)\cos(1000\pi t).$$

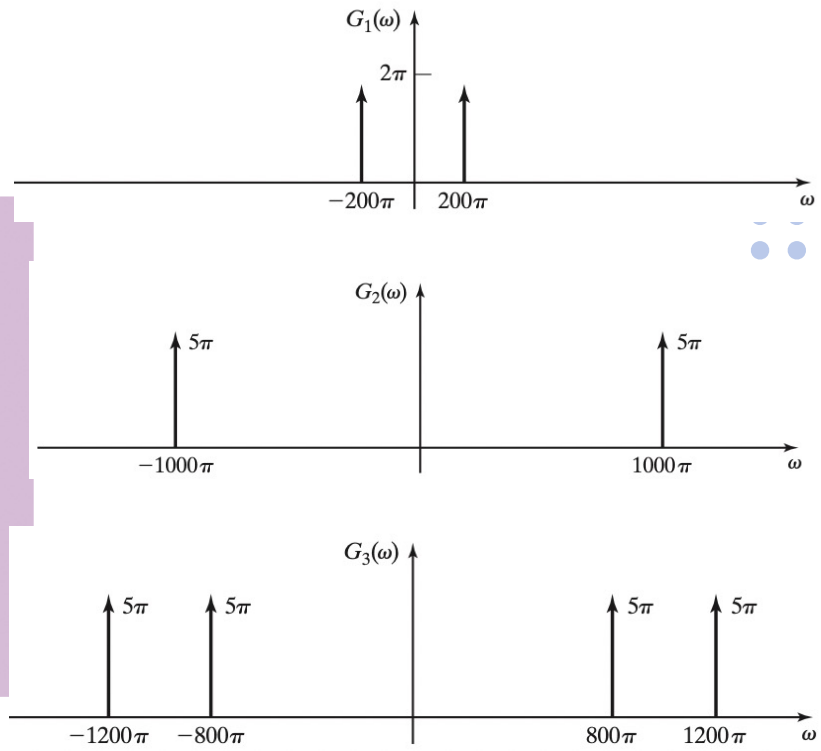
$$g_3(t) = 10\cos(200\pi t)\frac{e^{j1000\pi t} + e^{-j1000\pi t}}{2}$$

$$= 5\cos(200\pi t)e^{j1000\pi t} + 5\cos(200\pi t)e^{-j1000\pi t}.$$

$$G_3(\omega) = 5\pi[\delta(\omega - 200\pi - 1000\pi) + \delta(\omega + 200\pi - 1000\pi)]$$
$$+ 5\pi[\delta(\omega - 200\pi + 1000\pi) + \delta(\omega + 200\pi + 1000\pi)].$$

In final form, we write

$$G_3(\omega) = 5\pi[\delta(\omega - 1200\pi) + \delta(\omega - 800\pi) + \delta(\omega + 800\pi) + \delta(\omega + 1200\pi)].$$



$$g_3(t) = \mathcal{F}^{-1}\{5\pi[\delta(\omega - 1200\pi) + \delta(\omega + 1200\pi)]\}$$
$$+ \mathcal{F}^{-1}\{5\pi[\delta(\omega - 800\pi) + \delta(\omega + 800\pi)]\}$$
$$= 5\cos 1200\pi t + 5\cos 800\pi t.$$

The Fourier Transform 5.2: Properties of Fourier Transform

- **Symmetry:**

$$F(-\omega) = F^*(\omega), \text{ for real-valued } f(t).$$

$$F^*(\omega) = \int_{-\infty}^{\infty} [f(t)e^{-j\omega t}]^* dt = \int_{-\infty}^{\infty} f^*(t)e^{j\omega t}\, dt.$$

Because $f(t)$ is real, $f^*(t) = f(t)$, and therefore

$$F^*(\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t} dt = F(-\omega)$$

EXAMPLE 5.16 **The symmetry property of the Fourier transform**

Derive the Fourier transform of $x(t) = e^{2t}u(-t)$.

This result can also be easily derived using the Fourier integral:

$$X(\omega) = \int_{-\infty}^{\infty} e^{2t}u(-t)e^{-j\omega t} dt = \int_{-\infty}^{0} e^{(2-j\omega)t} dt = \frac{1}{2 - j\omega}$$

$$e^{-at}u(t),\ a > 0 = f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega) = \frac{1}{a + j\omega}.$$

$$X(\omega) = F^*(\omega) = F(-\omega) = \frac{1}{2 - j\omega}$$

The Fourier Transform 5.2: Properties of Fourier Transform

- **Time Integration and Differentiation:**

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)$$

- Time integration:

$$g(t) = \int_{-\infty}^{t} f(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega) = G(\omega)$$

- Time differentiation:

$$\frac{d[f(t)]}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega F(\omega)$$

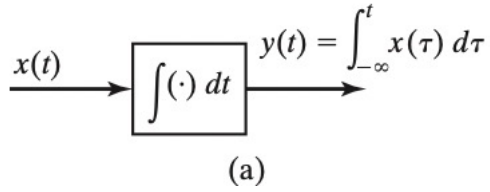
$$\frac{d^n[f(t)]}{dt^n} \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^n F(\omega)$$

The Fourier Transform 5.2: Properties of Fourier Transform

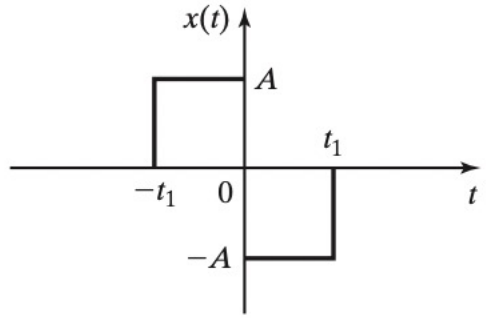
EXAMPLE 5.13 **The time-integration property of the Fourier transform**

Figure 5.1l(a) shows a linear system that consists of an integrator. As discussed in Section 1.2, this can be physically realized electronically by a combination of an operational amplifier, resistors, and capacitors. The input signal is a pair of rectangular pulses as shown in Figure 5.11(b). Using time-domain integration we can see that the output signal would be a triangular waveform, as shown in Figure 5.11(c). We wish to know the frequency spectrum of the output signal. We have not derived the Fourier transform of a triangular wave; however, we do know the Fourier transform of a rectangular pulse such as is present at the input of the system. Using the properties of linearity and time shifting, we can write the input signal as

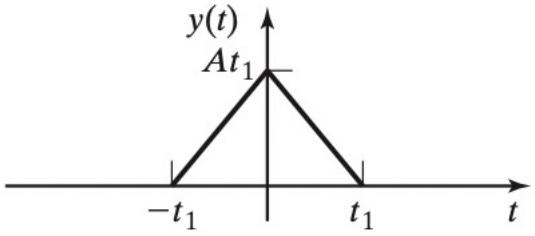
$$x(t) = A\,\text{rect}\left[\frac{t + t_1/2}{t_1}\right] - A\,\text{rect}\left[\frac{t - t_1/2}{t_1}\right]$$



(a)

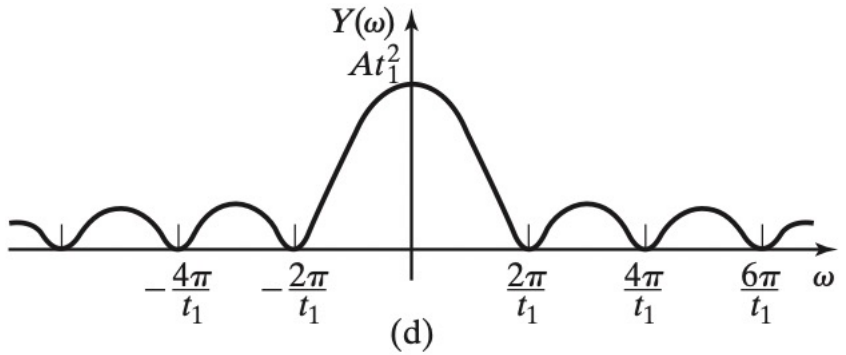


(b)



(c)

$$X(\omega) = At_1\,\text{sinc}(t_1\omega/2)[e^{j\omega t_1/2} - e^{-j\omega t_1/2}]$$
$$= 2jAt_1\,\text{sinc}(t_1\omega/2)\,\sin(t_1\omega/2)$$
$$= j\omega At_1^2\,\text{sinc}(t_1\omega/2)\left[\frac{\sin(t_1\omega/2)}{t_1\omega/2}\right]$$
$$= j\omega At_1^2\,\text{sinc}^2(t_1\omega/2).$$



(d)

$$\text{tri}(t/T) = \begin{cases} 1 - \dfrac{|t|}{T}, & |t| < T \\ 0, & |t| \geqq T \end{cases}$$

$$Y(\omega) = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

$$Y(\omega) = At_1^2\,\text{sinc}^2(t_1\omega/2).$$

$$\text{tri}(t/T) \overset{\mathcal{F}}{\longleftrightarrow} T\,\text{sinc}^2(T\omega/2)$$

The Fourier Transform 5.2: Properties of Fourier Transform

EXAMPLE 5.14 **Fourier transform of the signum function**

We now find the Fourier transform of the signum function shown i

$$f(t) = \text{sgn}(t).$$

$$\frac{d[f(t)]}{dt} = 2\delta(t)$$

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

$$j\omega F(\omega) = 2.$$

$$F(\omega) = \frac{2}{j\omega} + k\delta(\omega)$$

$$\text{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{j\omega}.$$

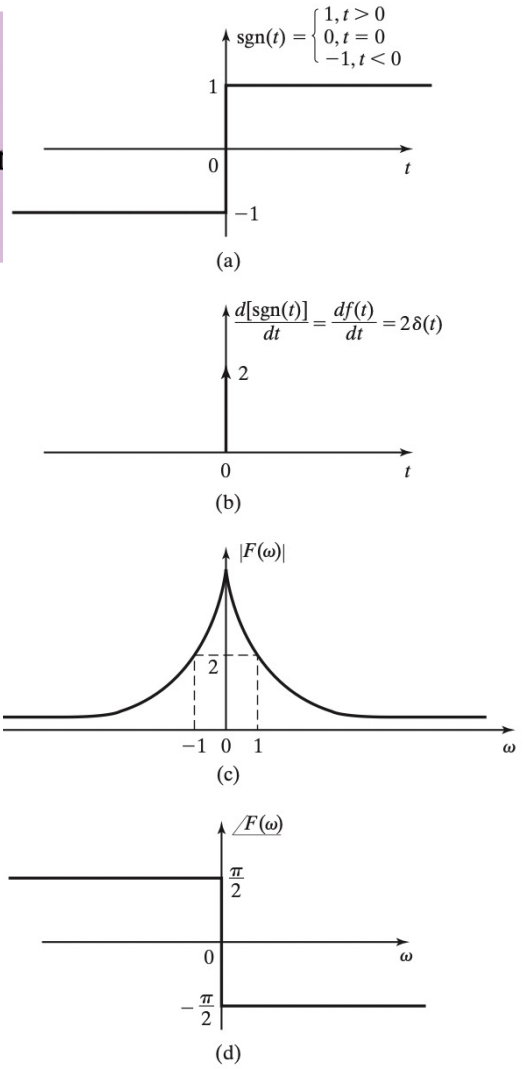


Figure 5.12 Finding the frequency spectrum of the signum function.

The Fourier Transform: Table 5.2

Time Domain Signal	Fourier Transform
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(\omega)$
$\delta(t)$	1
$A\delta(t-t_0)$	$Ae^{-j\omega t_0}$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
K	$2\pi K\delta(\omega)$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\text{rect}(t/T)$	$T\text{sinc}\,(\omega T/2)$
$\cos\,(\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$
$\text{rect}(t/T)\cos(\omega_0 t)$	$\frac{T}{2}\left[\text{sinc}\left(\frac{(\omega-\omega_0)T}{2}\right)+\text{sinc}\left(\frac{(\omega+\omega_0)T}{2}\right)\right]$
$\frac{\beta}{\pi}\text{sinc}(\beta t)$	$\text{rect}(\omega/2\beta)$
$\text{tri}(t/T)$	$T\,\text{sinc}^2(T\,\omega/2)$
$\text{sinc}^2(Tt/2)$	$\frac{2\pi}{T}\text{tri}(\omega/T)$
$e^{-at}u(t),\ \text{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
$te^{-at}u(t),\ \text{Re}\{a\}>0$	$\left(\frac{1}{a+j\omega}\right)^2$
$t^{n-1}e^{-at}u(t),\ \text{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
$e^{-a\lvert t\rvert},\ \text{Re}\{a\}>0$	$\frac{2a}{a^2+\omega^2}$
$\sum_{n=-\infty}^{\infty} g(t-nT_0)$	$\sum_{n=-\infty}^{\infty}\omega_0 G(n\omega_0)\delta(\omega-n\omega_0),\ \omega_0=\frac{2\pi}{T_0}$
$\sum_{n=-\infty}^{\infty} g(t-nT_0)=\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$	$2\pi\sum_{n=-\infty}^{\infty} C_n\delta(\omega-n\omega_0),\ \omega_0=\frac{2\pi}{T_0}$
$\delta_T(t)$	$\sum_{k=-\infty}^{\infty}\omega_0\delta(\omega-k\omega_0)$