الهيئة الملكية للجبيل وينبع
Royal Commission for Jubail & Yanbu

كلية الجبيل الصناعية
JUBAIL INDUSTRIAL COLLEGE

EEN314: Signals and Systems

Lecture 10

Lecture Outline

- Fourier Series 4.6: Fourier Series Transformations
- The Fourier Transform 5.1: Definition of the Fourier Transform
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Fourier Series 4.6: Fourier Series Transformations

- **Amplitude Transformation:**

$$x(t) = \sum_{k=-\infty}^{\infty} C_{kx}\, e^{jk\omega_0 t}$$

$$y(t) = Ax(t) + B = \sum_{k=-\infty}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

$$y(t) = B + A\sum_{k=-\infty}^{\infty} C_{kx}\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

$$= (B + AC_{0x}) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} A\, C_{kx}\; e^{jk\omega_0 t} = C_{0y} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{ky}\; e^{jk\omega_0 t}$$

- So,

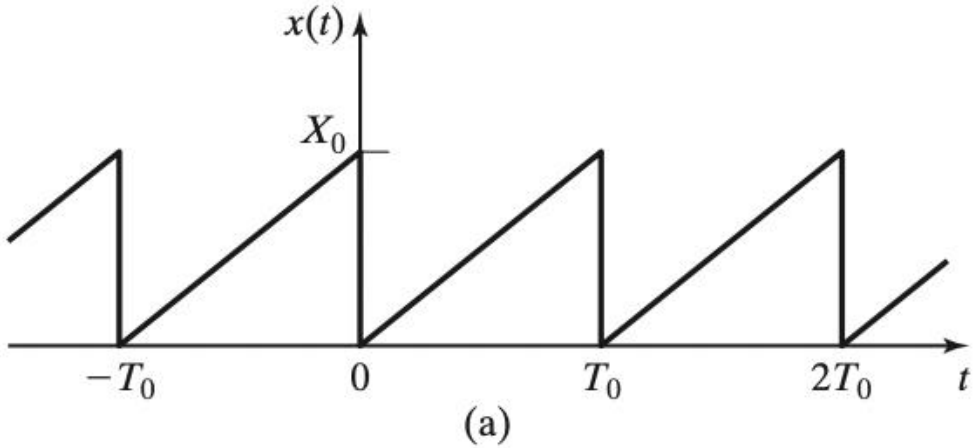
$$C_{0y} = B + AC_{0x} \qquad C_{ky} = AC_{kx} \text{ whenever } k \neq 0$$

Fourier Series 4.6: Fourier Series Transformations

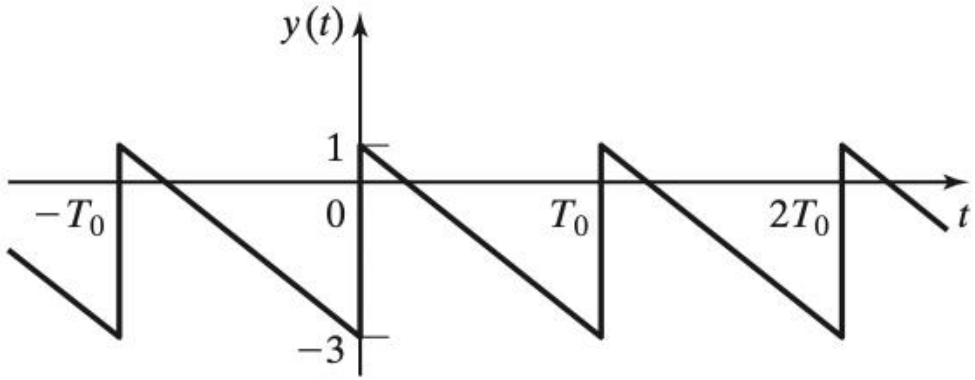
EXAMPLE 4.10 **Amplitude transformation for a Fourier series**

Consider the sawtooth signal $x(t)$ of Figure 4.23(a). From Table 4.3, the Fourier series is given by

$$x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{X_0}{2\pi k} e^{j\pi/2} e^{jk\omega_0 t}.$$



(a)



(b)

Find the exponential F.S. of $y(t)$.

$$y(t) = Ax(t) + B$$

$$C_{0y} = B + AC_{0x}$$

$$C_{ky} = AC_{kx} \text{ whenever } k \neq 0$$

$$y(t) = Ax(t) + B = -\frac{4}{X_0}x(t) + 1.$$

$$C_{0y} = AC_{0x} + B = \left(-\frac{4}{X_0}\right)\frac{X_0}{2} + 1 = -1$$

$$C_{ky} = AC_{kx} = \left(-\frac{4}{X_0}\right)\frac{X_0}{2\pi k} e^{j\pi/2} = \frac{2}{\pi k} e^{-j\pi/2}, \quad k \neq 0.$$

$$-j = -e^{j\pi/2} = e^{-j\pi/2}$$

$$y(t) = -1 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2}{\pi k} e^{-j\pi/2} e^{jk\omega_0 t}$$

Fourier Series 4.6:
Fourier Series Transformations

- **Time Transformation:**

$$y(t) = x(at + b)$$

- **Case 1:** (Time Reversal) ➔ $a = -1, b = 0$ ➔ $y(t) = x(-t)$

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} C_{kx}\, e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

$$= \sum_{k=\infty}^{-\infty} C_{-kx}\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

$$\therefore \quad C_{ky} = C_{-kx} = C_{kx}^*$$

- **Case 2:** (Time Shift) ➔ $a = 1, b = -t_o$ ➔ $y(t) = x(t - t_o)$

$$y(t) = x(t - t_o) = \sum_{k=-\infty}^{\infty} C_{kx}\, e^{jk\omega_0 (t - t_o)} = \sum_{k=-\infty}^{\infty} C_{kx}\, e^{jk\omega_0 t} e^{-jk\omega_0 t_o} = \sum_{k=-\infty}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

- So,

$$C_{ky} = C_{kx}\, e^{-jk\omega_0 t_o}$$

Fourier Series 4.6: Fourier Series Transformations

TABLE 4.8 Amplitude and Time Transformations

Amplitude	$y(t) = Ax(t) + B$
	$C_{0y} = AC_{0x} + B$
	$C_{ky} = AC_{kx}, k \neq 0$
Time	$\tau = -t \Rightarrow C_{ky} = C_{kx}^*$
	$\tau = t - t_0 \Rightarrow C_{ky} = C_{kx} e^{-jk\omega_0 t_0}$

Fourier Series 4.6: Fourier Series Transformations

- **Exercise:** Find the exponential F.S. of $y(t)$.

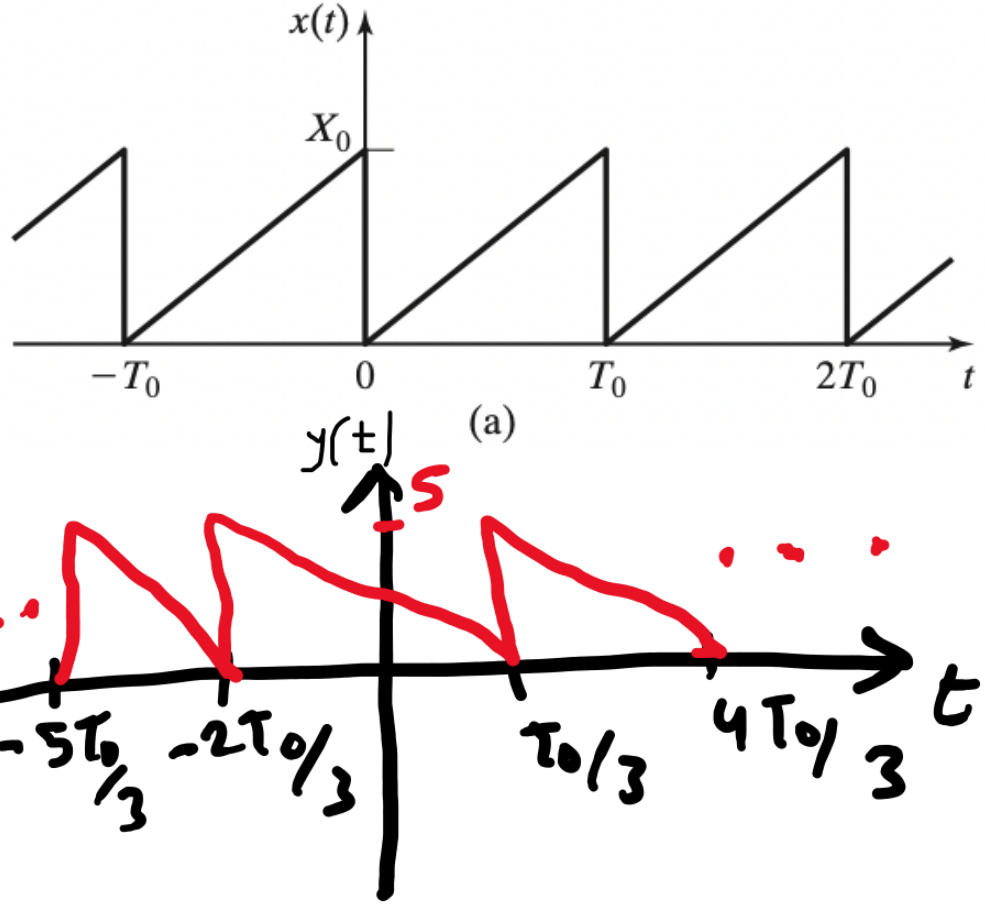
EXAMPLE 4.11 **Amplitude and time transformations for a Fourier series**

We again consider the sawtooth waveform of Example 4.10 and Figure 4.23. Hence,

$$x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{X_0}{2\pi k} e^{j\pi/2} e^{jk\omega_0 t}.$$

TABLE 4.8 Amplitude and Time Transformations

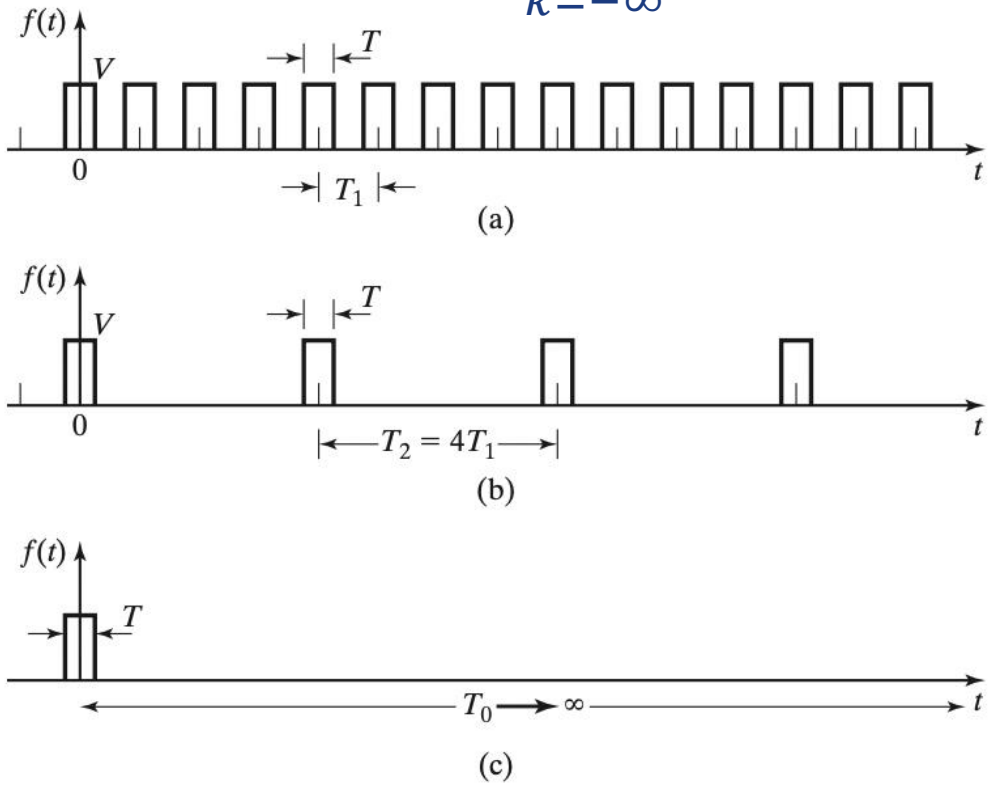
Amplitude	$y(t) = Ax(t) + B$
	$C_{0y} = AC_{0x} + B$
	$C_{ky} = AC_{kx}, k \neq 0$
Time	$\tau = -t \Rightarrow C_{ky} = C_{kx}^*$
	$\tau = t - t_0 \Rightarrow C_{ky} = C_{kx} e^{-jk\omega_0 t_0}$



The Fourier Transform 5.1: Definition of the Fourier Transform

- By using the Fourier series, any periodic function can be approximated.
- The exponential form is given by:

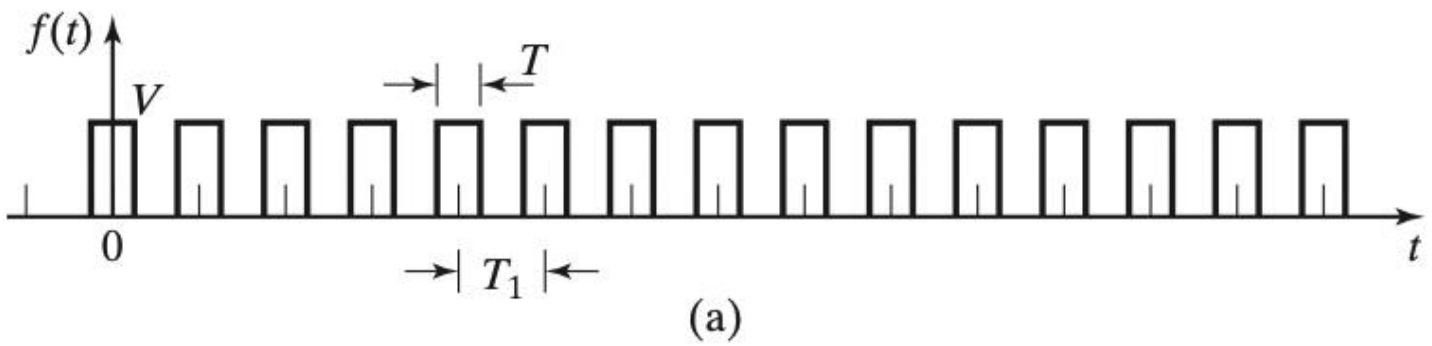
$$f(t) = \sum_{k=-\infty}^{\infty} C_k \, e^{jk\omega_0 t}, \qquad C_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{jk\omega_0 t}dt$$



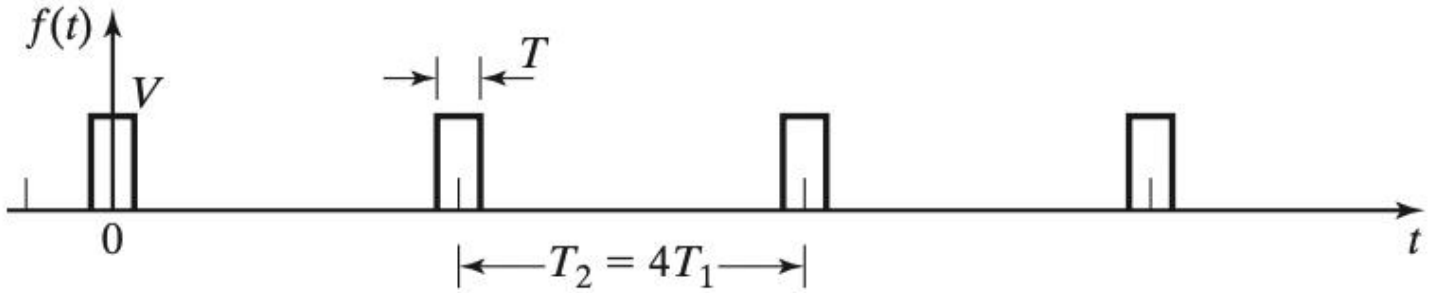
- It is known that $\omega_0 = \frac{2\pi}{T_0}$
- The spacing between the spectrum lines:
  - $\Delta\omega = (k+1)\omega_0 - k\omega_0 = \omega_0$
- As the T_0 increases, the ω_0 decreases.
- When $T_0 \to \infty$ ➔ $\Delta\omega \to d\omega$

$$\lim_{T_0\to\infty} \frac{2\pi}{T_0} = d\omega$$

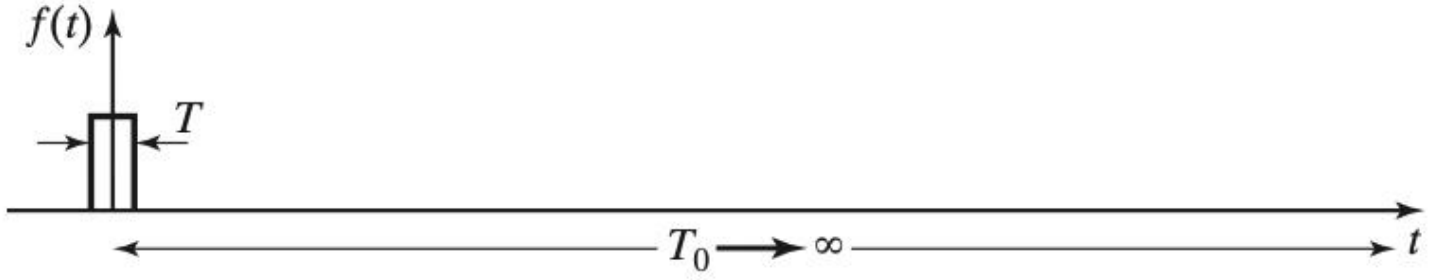
The Fourier Transform 5.1: Definition of the Fourier Transform



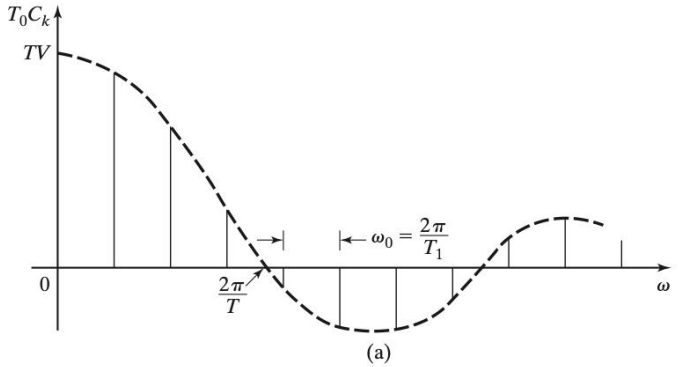
(a)



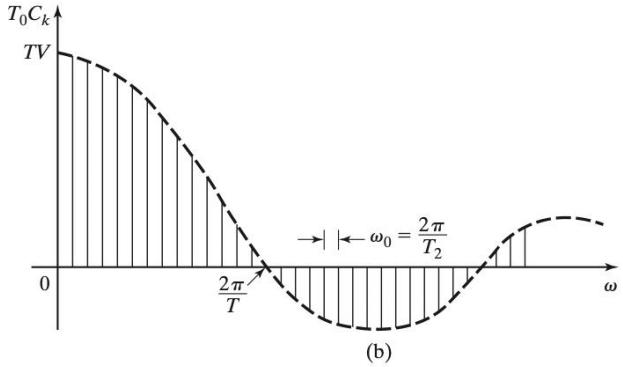
(b)



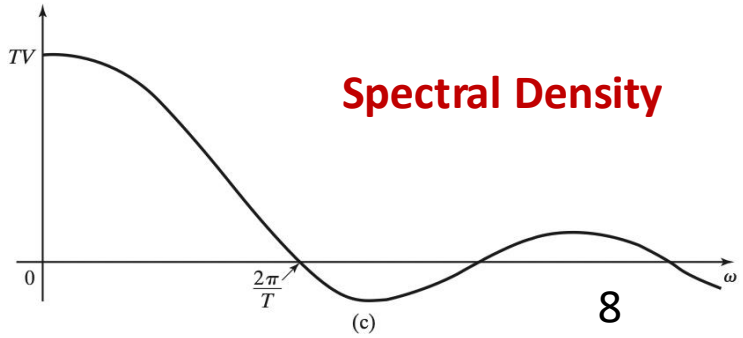
(c)



(a)



(b)



(c)

Spectral Density

The Fourier Transform 5.1: Definition of the Fourier Transform

- So, for $T_0 \to \infty$, the C_k will be represented as:

$$C_{k\infty} = \lim_{T_0 \to \infty} \frac{1}{2\pi} \frac{2\pi}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-(jk2\pi/T_0)t}\, dt = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \right] d\omega.$$

- The Fourier transform is ➔ $$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt.$$

- By applying it to $f(t)$ ➔ $$f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} F(\omega) d\omega e^{jk\omega_0 t} := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F(\omega) e^{jk\omega_0 t} d\omega.$$

- The summation becomes an integral and $\lim_{T_0 \to \infty} k\,\omega_0 = \omega$, the equation can be rewritten as:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega = F^{-1}\{F(\omega)\}.$$

The Fourier Transform 5.1: Definition of the Fourier Transform

- Thus, the Fourier transform (FT) is defined by:

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

- The inverse Fourier transform (IFT) is defined by:

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega.$$

- They are called a transform pair where the mathematical notation is represented by:

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega).$$

- As you can notice, it is used for aperiodic signals.

The Fourier Transform 5.1: Definition of the Fourier Transform

- The FT of $f(t) = V\ rect(\frac{t}{T})$ is given below:

$$F(\omega) = \int_{-T/2}^{+T/2} Ve^{-j\omega t}\,dt = V\left[\frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T/2}^{+T/2}\right]$$

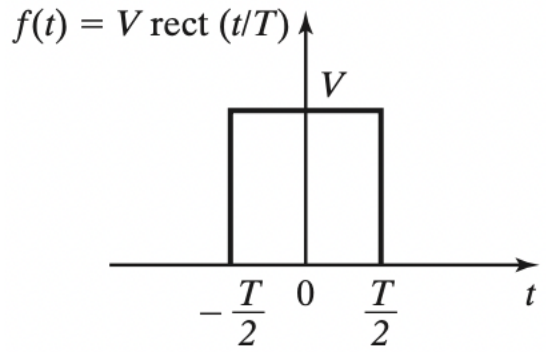
$$= V\left[\frac{e^{-jT\omega/2} - e^{+jT\omega/2}}{-j\omega}\right] = \frac{TV}{\omega T/2}\left[\frac{e^{jT\omega/2} - e^{-jT\omega/2}}{j2}\right]$$

$$= TV\left[\frac{\sin(T\omega/2)}{T\omega/2}\right] := TV\,\mathrm{sinc}(T\omega/2).$$

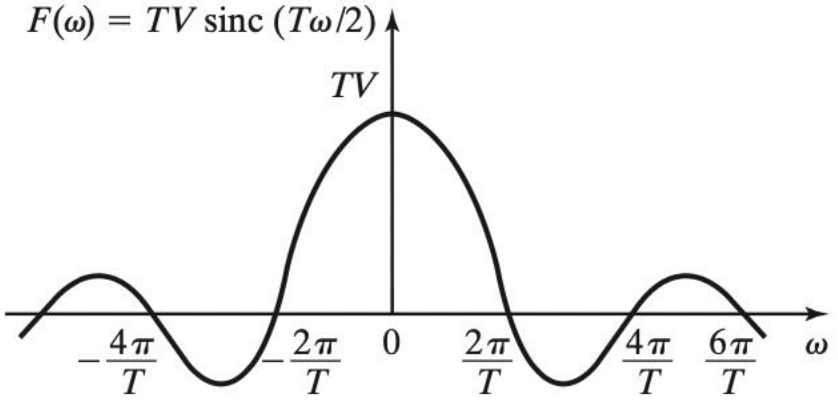
$$\mathcal{F}\{V[u(t + T/2) - u(t - T/2)]\} = TV\,\mathrm{sinc}(T\omega/2).$$

For $\mathrm{rect}(t/T) = [u(t + T/2) - u(t - T/2)].$

The transform pairs is $\mathrm{rect}(t/T) \overset{\mathcal{F}}{\longleftrightarrow} T\,\mathrm{sinc}(T\omega/2)$



$$f(t) = Vu(t + T/2) - Vu(t - T/2).$$



The Fourier Transform 5.1: Definition of the Fourier Transform

- **Existence:**
- It is clear that not all signals (functions) have Fourier transform.
- Sufficient conditions for Fourier transform are of Dirichlet conditions:

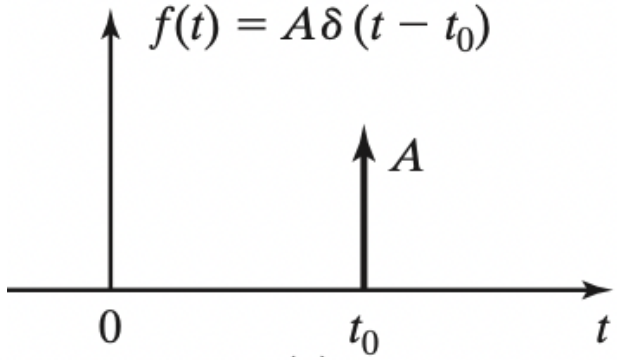
- On any finite interval:
  - $f(t)$ is bounded,
  - $f(t)$ has a finite number of maxima and minima, and
  - $f(t)$ has a finite number of discontinuities.

- $f(t)$ be absolutely integrable
  - $\int_{-\infty}^{\infty} |f(t)| dt = M < \infty$

The Fourier Transform 5.1: Definition of the Fourier Transform

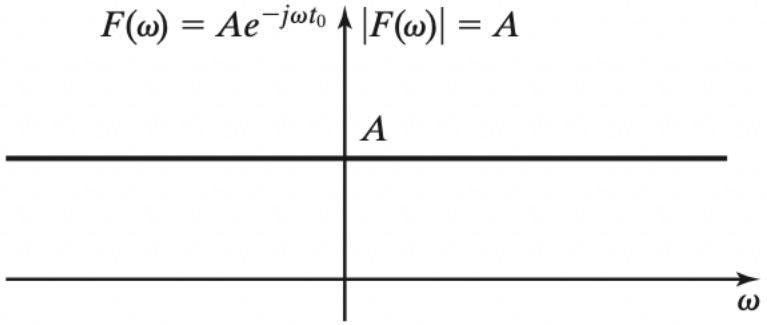
- Consider the impulse function in time domain:

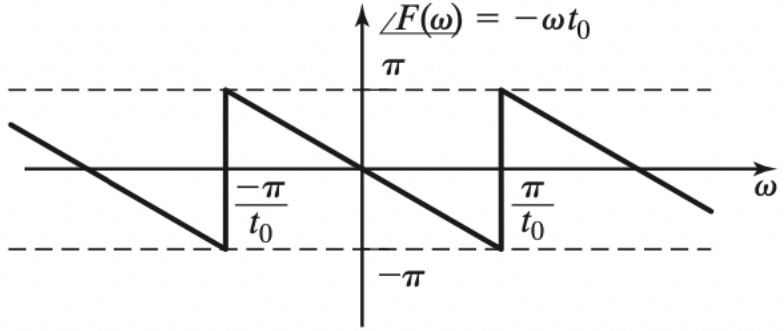
$$f(t) = A\delta(t - t_0),$$



- By applying the FT:

$$F(\omega) = \mathcal{F}\{A\delta(t - t_0)\} = \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-j\omega t}\,dt. \quad \Rightarrow \quad \mathcal{F}\{A\delta(t - t_0)\} = Ae^{-j\omega t_0}.$$





- So, for impulse function with $t_0 = 0$ and $A = 1$:

$$f(t) = \delta(t). \quad \Rightarrow \quad \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

The Fourier Transform 5.1: Definition of the Fourier Transform

$$\delta(\omega-\omega_0)=\begin{cases}\text{undefined}, & \omega=\omega_0\\ 0, & \omega\neq\omega_0\end{cases},$$

$$\int_{-\infty}^{\infty}\delta(\omega-\omega_0)d\omega=1,$$

$$F(\omega)\delta(\omega-\omega_0)=F(\omega_0)\delta(\omega-\omega_0),$$

$$\int_{-\infty}^{\infty}F(\omega)\delta(\omega-\omega_0)d\omega=F(\omega_0),\ \text{etc.}$$

- Consider the impulse function in frequency domain:

$$F(\omega)=\delta(\omega-\omega_0)$$

- By applying the IFT:

$$f(t)=\mathcal{F}^{-1}\{F(\omega)\}=\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega \quad\Rightarrow\quad f(t)=\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\}=\frac{1}{2\pi}e^{j\omega_0 t}$$

- So, the fourier pair transform: $e^{j\omega_0 t}\overset{\mathcal{F}}{\longleftrightarrow}2\pi\delta(\omega-\omega_0)$

