1

First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Example 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we

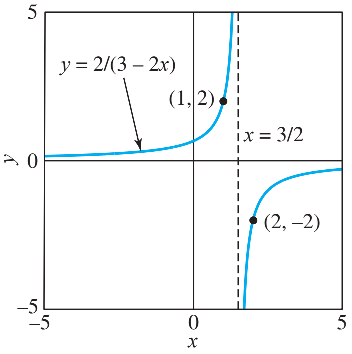

FIGURE 1.1.7. The solutions of $y' = y^2$ defined by $y(x) = 2/(3 - 2x)$.

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3 - 2x)$. The left-hand branch is the graph on $(-\infty, \frac{3}{2})$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $(\frac{3}{2}, \infty)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1 + x^2}$
10. $x^2y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x\sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that y(x) satisfies the given differential equation. Then determine a value of the constant C so that y(x) satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$. Can you *guess* what the graph of such a function g might look like?
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

Differential Equations as Models

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.
34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

Q3 // $y'' + 4y = 0$

$\boxed{y_1 = \cos 2x}$ ✓ $\boxed{y_2 = \sin 2x}$

$y_1' = -2\sin 2x$ $y_2' = 2\cos 2x$

$\boxed{y_1'' = -4\cos 2x}$ $\boxed{y_2'' = -4\sin 2x}$

$\Rightarrow -4\cos 2x + 4\cos 2x = 0$ ✓

$-4\sin 2x + 4\sin 2x = 0$ ✓

$$\left.\begin{aligned} y_1 &= \cos 2x \\ y_2 &= \sin 2x \end{aligned}\right\} ✓ \quad \boxed{y'' + 4y = 0}$$

Q6 $y'' + \boxed{4y'} + 4y = \boxed{0}$ ← $x = 1$

$y_1 = e^{-2x}$ ✓ $\qquad y_2 = x \cdot e^{-2x}$ ✓

$y_1' = -2e^{-2x}$ $\qquad y_2' = \boxed{1e^{-2x} - 2xe^{-2x}}$

$y_1'' = +4e^{-2x}$ $\qquad y_2'' = -2e^{-2x}$

$-2[1e^{-2x} - 2xe^{-2x}]$

$4e^{-2x} - 8e^{-2x} + 4e^{-2x}$

$8 - 8 = 0$

✓

$y_2'' = -2e^{-2x} - 2e^{-2x} + 4xe^{-2x}$

$\cancel{-4e^{-2x}} + \cancel{4e^{-2x}} + \cancel{4e^{-2x}} - \cancel{8xe^{-2x}} + \cancel{4xe^{-2x}}$

$+ \cancel{8xe^{-2x}}$

$= 0$

Q7 $y'' - 2y' + 2y = \boxed{0}$

$y_1 = e^x \cos x$

$y_2 = e^x \sin x$

$$\boxed{y_1' = e^x \cos x - e^x \sin x}$$

$$y_1'' = \cancel{e^x \cos x} - e^x \sin x - e^x \sin x - \cancel{e^x \cos x}$$

$$\boxed{y_1'' = -2 e^x \sin x}$$

$$\cancel{-2e^x \sin x} - \cancel{2e^x \cos x} + \cancel{2e^x \sin x} + \cancel{2e^x \cos x}$$

$$= \boxed{0}$$

⑩ $x^2y'' + xy' - y = \boxed{\ln x}$??

$y_1 = x - \ln x$ $\qquad$ $y_2 = \frac{1}{x} - \ln x$

$y_1' = 1 - \frac{1}{x}$ ✓ $\qquad$ $y_2' = \frac{-1}{x^2} - \frac{1}{x}$

$y_1'' = +\frac{1}{x^2}$ $\qquad$ $y_2'' = \frac{2}{x^3} + \frac{1}{x^2}$

$\cancel{x^2} \cdot \frac{1}{\cancel{x^2}} + \cancel{x}\left[1 - \frac{1}{\cancel{x}}\right] - \left[x - \ln x\right] = \ln x$

$\cancel{1} + \cancel{x} - \cancel{1} - \cancel{x} + \boxed{\ln x} = \ln x$

$\cancel{x^2}\left[\frac{2}{x^3} + \frac{1}{\cancel{x^2}}\right] + \cancel{x}\left[\frac{-1}{x^2} - \frac{1}{\cancel{x}}\right] - \left[\frac{1}{x} - \ln x\right]$

$= \frac{2}{x} \cancel{+1} - \frac{1}{x} \cancel{-1} - \frac{1}{x} + \boxed{\ln x}$

$\cancel{\frac{2}{x}} - \cancel{\frac{2}{x}}$

Q_{12} $x^2y'' - xy' + 2y = \boxed{0}$?

$\boxed{y_1 = x\cdot\cos(\ln x)}$ $y_2 = x\sin(\ln x)$

$y_1' = \cos(\ln x) - \cancel{x}\sin(\ln x)\cdot\frac{1}{\cancel{x}}$

$\boxed{y_1' = \cos(\ln x) - \sin(\ln x)}$

$\boxed{y_1'' = -\sin(\ln x)\cdot\frac{1}{x} - \cos(\ln x)\cdot\frac{1}{x}}$

$x^{2}\left[-\frac{\sin(\ln x)}{\cancel{x}} - \frac{\cos(\ln x)}{\cancel{x}}\right] - x\left[\cos(\ln x) - \sin(\ln x)\right]$

$+ 2\left[x\cos(\ln x)\right]$

$$\left.\begin{array}{l} \boxed{\cancel{-x\sin(\ln x)}} - x\cos(\ln x) \\ \boxed{\cancel{+x\sin(\ln x)}} - x\cos(\ln x) \end{array}\right\} -2x$$

$+2x\cos(\ln x) = 0$

Q16 $\quad 3y'' + 3y' - 4y = 0$

$y = e^{rx} \rightarrow y' = re^{rx} \rightarrow y'' = re^{2rx}$

$3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0$

$\underbrace{e^{rx}}_{e^{rx} \neq 0} \underbrace{[3r^2 + 3r - 4]}_{\overset{a}{3}r^2 + \overset{b}{3}r \overset{c}{-4} = 0} = 0$

$$r_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$r_{1,2} = \frac{-3 \mp \sqrt{9 + 4(3)(\cancel{-}4)}}{2(3)} \quad \underbrace{}_{} \rightarrow 48$$

$$\boxed{r_{1,2} = \frac{-3 \mp \sqrt{57}}{6}}$$

Q1a $y' = y + 1$

$$\boxed{y(x) = Ce^x - 1}$$

$$\boxed{y(0) = 5}$$

$x \quad y$

$$\boxed{y' = Ce^x}$$

$$Ce^x = Ce^x \cancel{-1} \cancel{+1}$$

$$\boxed{e^0 = 1}$$

$$5 = C\cancel{e^{0}}^{1} - 1$$

$$5 = C - 1$$

$+1$

$$\Longrightarrow \boxed{C = 6}$$

$$\boxed{y(x) = 6e^x - 1}$$

Q28 $y' + y\tan x = \boxed{\cos x}$

$$\boxed{y(x) = (x+c)\cos x} \leftarrow y(\pi) = 0$$

x

y

$$\boxed{y' = \cos x - (x+c)\sin x}$$

$$\cos x - (x+c)\sin x + (x+c)\cancel{\cos x}\tan x \qquad \frac{\sin x}{\cancel{\cos x}}$$

$$\cos x \cancel{-(x+c)\sin x} \cancel{+(x+c)\sin x}$$

$$0 = (\pi + c)\cdot\cos\pi = -1 \qquad \cos\pi = -1$$

$$0 = -\pi - c \Longrightarrow \boxed{c = -\pi}$$

$+c$

$$\boxed{y = (x-\pi)\cos x}$$