9–14 Graph each function over the specified interval. Then use simple area formulas from geometry to find the area function $A(x)$ that gives the area between the graph of the specified function f and the interval $[a, x]$. Confirm that $A'(x) = f(x)$ in every case. ■

9. $f(x) = 3;\ [a, x] = [1, x]$

10. $f(x) = 5;\ [a, x] = [2, x]$

11. $f(x) = 2x + 2;\ [a, x] = [0, x]$

12. $f(x) = 3x - 3;\ [a, x] = [1, x]$

13. $f(x) = 2x + 2;\ [a, x] = [1, x]$

14. $f(x) = 3x - 3;\ [a, x] = [2, x]$

15–18 True–False Determine whether the statement is true or false. Explain your answer. ■

15. If $A(n)$ denotes the area of a regular n-sided polygon inscribed in a circle of radius 2, then $\lim_{n\to+\infty} A(n) = 2\pi$.

16. If the area under the curve $y = x^2$ over an interval is approximated by the total area of a collection of rectangles, the approximation will be too large.

17. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A'(x) = f(x)$.

18. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A(x)$ will be a continuous function.

FOCUS ON CONCEPTS

19. Explain how to use the formula for $A(x)$ found in the solution to Example 2 to determine the area between the graph of $y = x^2$ and the interval $[3, 6]$.

20. Repeat Exercise 19 for the interval $[-3, 9]$.

21. Let A denote the area between the graph of $f(x) = \sqrt{x}$ and the interval $[0, 1]$, and let B denote the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$. Explain geometrically why $A + B = 1$.

22. Let A denote the area between the graph of $f(x) = 1/x$ and the interval $[1, 2]$, and let B denote the area between the graph of f and the interval $\left[\frac{1}{2}, 1\right]$. Explain geometrically why $A = B$.

23–24 The area $A(x)$ under the graph of f and over the interval $[a, x]$ is given. Find the function f and the value of a. ■

23. $A(x) = x^2 - 4$

24. $A(x) = x^2 - x$

25. Writing Compare and contrast the rectangle method and the antiderivative method.

26. Writing Suppose that f is a nonnegative continuous function on an interval $[a, b]$ and that $g(x) = f(x) + C$, where C is a positive constant. What will be the area of the region *between* the graphs of f and g?

✓**QUICK CHECK ANSWERS 4.1** 1. (a) $\frac{\pi}{2}$ (b) $1 + \frac{\sqrt{3}}{2}$ 2. 2 3. 9 4. $A(x) = \frac{x^2}{2};\ A'(x) = \frac{2x}{2} = x = f(x)$ 5. $\cos x + 1$

4.2 THE INDEFINITE INTEGRAL

In the last section we saw how antidifferentiation could be used to find exact areas. In this section we will develop some fundamental results about antidifferentiation.

ANTIDERIVATIVES

4.2.1 DEFINITION A function F is called an ***antiderivative*** of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,

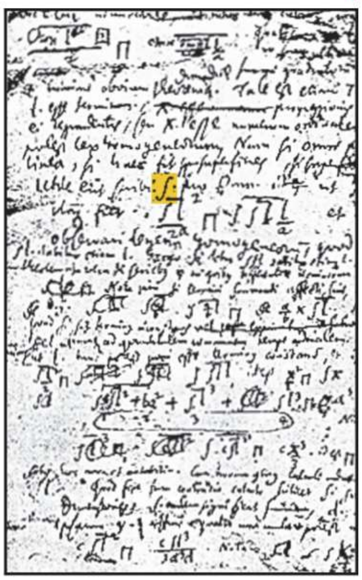
$$\tfrac{1}{3}x^3, \quad \tfrac{1}{3}x^3 + 2, \quad \tfrac{1}{3}x^3 - 5, \quad \tfrac{1}{3}x^3 + \sqrt{2}$$

are all antiderivatives of $f(x) = x^2$.

It is reasonable to ask if there are antiderivatives of a function f that cannot be obtained by adding some constant to a known antiderivative F. The answer is *no* since Theorem 3.8.3 tells us that if two functions have the same derivative on an open interval, then the functions differ by a constant on the interval. The following theorem summarizes these observations.

4.2.2 THEOREM *If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x) + C$ by choosing the constant C appropriately.*

THE INDEFINITE INTEGRAL



Reproduced from C. I. Gerhardt's "Briefwechsel von G. W. Leibniz mit Mathematikern (1899)".

Extract from the manuscript of Leibniz dated October 29, 1675 in which the integral sign first appeared (see yellow highlight).

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (1) is recast using ***integral notation***,

$$\int f(x)\,dx = F(x) + C \tag{2}$$

where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus,

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x) \tag{3}$$

The expression $\int f(x)\,dx$ is called an ***indefinite integral***. The adjective "indefinite" emphasizes that the result of antidifferentiation is a "generic" function, described only up to a constant term. The "elongated s" that appears on the left side of (2) is called an ***integral sign***,* the function $f(x)$ is called the ***integrand***, and the constant C is called the ***constant of integration***. Equation (2) should be read as:

The integral of $f(x)$ with respect to x is equal to $F(x)$ plus a constant.

The differential symbol, dx, in the differentiation and antidifferentiation operations

$$\frac{d}{dx}[\ ] \quad \text{and} \quad \int [\ ]\,dx$$

*This notation was devised by Leibniz. In his early papers Leibniz used the notation "omn." (an abbreviation for the Latin word "omnes") to denote integration. Then on October 29, 1675 he wrote, "It will be useful to write $\int$ for omn., thus $\int l$ for omn. l . . . ." Two or three weeks later he refined the notation further and wrote $\int [\ ]\,dx$ rather than $\int$ alone. This notation is so useful and so powerful that its development by Leibniz must be regarded as a major milestone in the history of mathematics and science.

serves to identify the independent variable. If an independent variable other than x is used, say t, then the notation must be adjusted appropriately. Thus,

$$\frac{d}{dt}[F(t)] = f(t) \quad \text{and} \quad \int f(t)\,dt = F(t) + C$$

are equivalent statements. For simplicity, the dx is sometimes absorbed into the integrand. For example,

$$\int 1\,dx \quad \text{can be written as} \quad \int dx$$

$$\int \frac{1}{x^2}\,dx \quad \text{can be written as} \quad \int \frac{dx}{x^2}$$

INTEGRATION FORMULAS

Integration is essentially educated guesswork—given the derivative f of a function F, one tries to guess what the function F is. However, many basic integration formulas can be obtained directly from their companion differentiation formulas. Some of the most important are given in Table 4.2.1.

Table 4.2.1
INTEGRATION FORMULAS

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$	5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \ (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \ (r \neq -1)$	6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

Example 1 The second integration formula in Table 4.2.1 will be easier to remember if you express it in words:

To integrate a power of x (other than -1), add 1 to the exponent and divide by the new exponent.

Formula 2 in Table 4.2.1 is not applicable to integrating x^{-1}; we will see how to integrate this function in Chapter 6.

Here are some examples:

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad r = 2$$

$$\int x^3\,dx = \frac{x^4}{4} + C \qquad r = 3$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C \qquad r = -5$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C = \tfrac{2}{3}(\sqrt{x})^3 + C \qquad r = \tfrac{1}{2}$$ ◀

PROPERTIES OF THE INDEFINITE INTEGRAL

Our first properties of antiderivatives follow directly from the simple constant factor, sum, and difference rules for derivatives.

4.2.3 THEOREM *Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then:*

(*a*) *A constant factor can be moved through an integral sign; that is,*

$$\int cf(x)\,dx = cF(x) + C$$

(*b*) *An antiderivative of a sum is the sum of the antiderivatives; that is,*

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(*c*) *An antiderivative of a difference is the difference of the antiderivatives; that is,*

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

PROOF In general, to establish the validity of an equation of the form

$$\int h(x)\,dx = H(x) + C$$

one must show that

$$\frac{d}{dx}[H(x)] = h(x)$$

We are given that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, so we know that

$$\frac{d}{dx}[F(x)] = f(x) \quad \text{and} \quad \frac{d}{dx}[G(x)] = g(x)$$

Thus,

$$\frac{d}{dx}[cF(x)] = c\frac{d}{dx}[F(x)] = cf(x)$$

$$\frac{d}{dx}[F(x) + G(x)] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[G(x)] = f(x) + g(x)$$

$$\frac{d}{dx}[F(x) - G(x)] = \frac{d}{dx}[F(x)] - \frac{d}{dx}[G(x)] = f(x) - g(x)$$

which proves the three statements of the theorem. ■

The statements in Theorem 4.2.3 can be summarized by the following formulas:

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{4}$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{5}$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx \tag{6}$$

However, these equations must be applied carefully to avoid errors and unnecessary complexities arising from the constants of integration. For example, if you use (4) to integrate $2x$ by writing

$$\int 2x\,dx = 2\int x\,dx = 2\left(\frac{x^2}{2} + C\right) = x^2 + 2C$$

then you will have an unnecessarily complicated form of the arbitrary constant. This kind of problem can be avoided by inserting the constant of integration in the final result rather than in intermediate calculations.

▶ Example 2 Evaluate

$$\text{(a)} \int 4\cos x\,dx \qquad \text{(b)} \int (x+x^2)\,dx$$

Solution (a). Since $F(x) = \sin x$ is an antiderivative for $f(x) = \cos x$ (Table 4.2.1), we obtain

$$\int 4\cos x\,dx = 4\int \cos x\,dx = 4\sin x + C$$

(4)

Solution (b). From Table 4.2.1 we obtain

$$\int (x+x^2)\,dx = \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C \blacktriangleleft$$

(5)

Parts (*b*) and (*c*) of Theorem 4.2.3 can be extended to more than two functions, which in combination with part (*a*) results in the following general formula:

$$\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\,dx = c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx + \cdots + c_n \int f_n(x)\,dx \qquad (7)$$

▶ Example 3

$$\int (3x^6 - 2x^2 + 7x + 1)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C \blacktriangleleft$$

Sometimes it is useful to rewrite an integrand in a different form before performing the integration. This is illustrated in the following example.

▶ Example 4 Evaluate

$$\text{(a)} \int \frac{\cos x}{\sin^2 x}\,dx \qquad \text{(b)} \int \frac{t^2 - 2t^4}{t^4}\,dt$$

Solution (a).

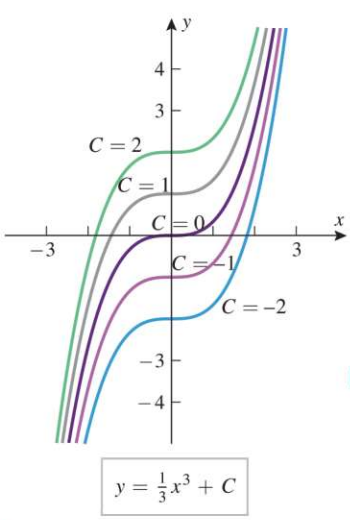
$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\,\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx = -\csc x + C$$

Formula 8 in Table 4.2.1

Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right) dt = \int (t^{-2} - 2)\,dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$



▲ **Figure 4.2.1**

INTEGRAL CURVES

Graphs of antiderivatives of a function f are called ***integral curves*** of f. We know from Theorem 4.2.2 that if $y = F(x)$ is any integral curve of $f(x)$, then all other integral curves are vertical translations of this curve, since they have equations of the form $y = F(x) + C$. For example, $y = \frac{1}{3}x^3$ is one integral curve for $f(x) = x^2$, so all the other integral curves have equations of the form $y = \frac{1}{3}x^3 + C$; conversely, the graph of any equation of this form is an integral curve (Figure 4.2.1).

In many problems one is interested in finding a function whose derivative satisfies specified conditions. The following example illustrates a geometric problem of this type.

▶ **Example 5** Suppose that a curve $y = f(x)$ in the xy-plane has the property that at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve given that it passes through the point $(2, 1)$.

Solution. Since the slope of the line tangent to $y = f(x)$ is dy/dx, we have $dy/dx = x^2$, and

$$y = \int x^2\,dx = \tfrac{1}{3}x^3 + C$$

Since the curve passes through $(2, 1)$, a specific value for C can be found by using the fact that $y = 1$ if $x = 2$. Substituting these values in the above equation yields

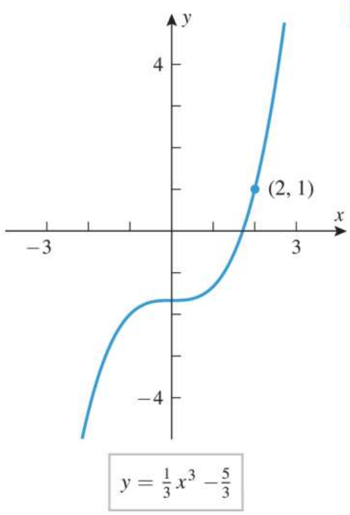
$$1 = \tfrac{1}{3}(2^3) + C \quad \text{or} \quad C = -\tfrac{5}{3}$$

so an equation of the curve is

$$y = \tfrac{1}{3}x^3 - \tfrac{5}{3}$$

(Figure 4.2.2). ◀

In Example 5, the requirement that the graph of f pass through the point $(2, 1)$ selects the single integral curve $y = \frac{1}{3}x^3 - \frac{5}{3}$ from the family of curves $y = \frac{1}{3}x^3 + C$ (Figure 4.2.2).



▲ **Figure 4.2.2**

INTEGRATION FROM THE VIEWPOINT OF DIFFERENTIAL EQUATIONS

We will now consider another way of looking at integration that will be useful in our later work. Suppose that $f(x)$ is a known function and we are interested in finding a function $F(x)$ such that $y = F(x)$ satisfies the equation

$$\frac{dy}{dx} = f(x) \tag{8}$$

The solutions of this equation are the antiderivatives of $f(x)$, and we know that these can be obtained by integrating $f(x)$. For example, the solutions of the equation

$$\frac{dy}{dx} = x^2 \tag{9}$$

are

$$y = \int x^2\,dx = \frac{x^3}{3} + C$$

Equation (8) is called a ***differential equation*** because it involves a derivative of an unknown function. Differential equations are different from the kinds of equations we have encountered so far in that the unknown is a *function* and not a *number* as in an equation such as $x^2 + 5x - 6 = 0$.

Sometimes we will not be interested in finding all of the solutions of (8), but rather we will want only the solution whose graph passes through a specified point (x_0, y_0). For example, in Example 5 we solved (9) for the integral curve that passed through the point $(2, 1)$.

For simplicity, it is common in the study of differential equations to denote a solution of $dy/dx = f(x)$ as $y(x)$ rather than $F(x)$, as earlier. With this notation, the problem of finding a function $y(x)$ whose derivative is $f(x)$ and whose graph passes through the point (x_0, y_0) is expressed as

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{10}$$

This is called an ***initial-value problem***, and the requirement that $y(x_0) = y_0$ is called the ***initial condition*** for the problem.

▶ **Example 6** Solve the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

Solution. The solution of the differential equation is

$$y = \int \cos x\,dx = \sin x + C \tag{11}$$

The initial condition $y(0) = 1$ implies that $y = 1$ if $x = 0$; substituting these values in (11) yields

$$1 = \sin(0) + C \quad \text{or} \quad C = 1$$

Thus, the solution of the initial-value problem is $y = \sin x + 1$. ◀

SLOPE FIELDS

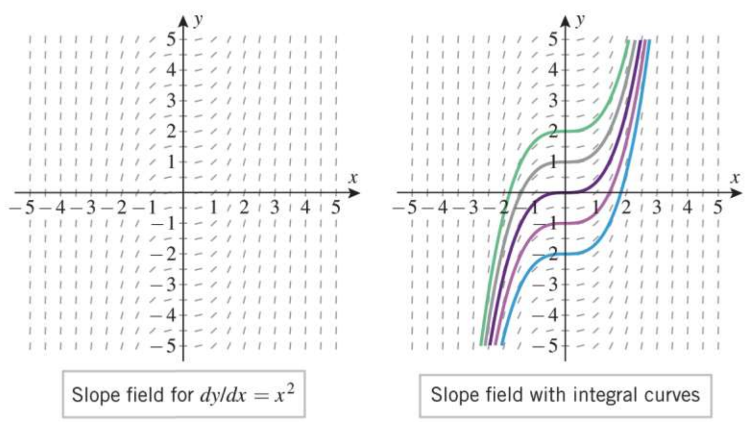
If we interpret dy/dx as the slope of a tangent line, then at a point (x, y) on an integral curve of the equation $dy/dx = f(x)$, the slope of the tangent line is $f(x)$. What is interesting about this is that the slopes of the tangent lines to the integral curves can be obtained without actually solving the differential equation. For example, if

$$\frac{dy}{dx} = \sqrt{x^2 + 1}$$

then we know without solving the equation that at the point where $x = 1$ the tangent line to an integral curve has slope $\sqrt{1^2 + 1} = \sqrt{2}$; and more generally, at a point where $x = a$, the tangent line to an integral curve has slope $\sqrt{a^2 + 1}$.

A geometric description of the integral curves of a differential equation $dy/dx = f(x)$ can be obtained by choosing a rectangular grid of points in the xy-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small portions of the tangent lines through those points. The resulting picture, which is called a ***slope field*** or ***direction field*** for the equation, shows the "direction" of the integral curves at the gridpoints. With sufficiently many gridpoints it is often possible to visualize the integral curves themselves; for example, Figure 4.2.3*a* shows a slope field for the differential equation $dy/dx = x^2$, and Figure 4.2.3*b* shows that same field with the integral curves imposed on it—the more gridpoints that are used, the more completely the slope field reveals the shape of the integral curves. However, the amount of computation can be considerable, so computers are usually used when slope fields with many gridpoints are needed.

Slope fields will be studied in more detail later in the text.



▶ **Figure 4.2.3** (*a*) (*b*)

✓QUICK CHECK EXERCISES 4.2 *(See page 217 for answers.)*

1. A function F is an antiderivative of a function f on an interval if ________ for all x in the interval.

2. Write an equivalent integration formula for each given derivative formula.

   (a) $\dfrac{d}{dx}[\sqrt{x}] = \dfrac{1}{2\sqrt{x}}$ (b) $\dfrac{d}{dx}[\sin x] = \cos x$

3. Evaluate the integrals.

   (a) $\displaystyle\int [x^3 + x + 5]\, dx$ (b) $\displaystyle\int [\sec^2 x - \csc x \cot x]\, dx$

4. The graph of $y = x^2 + x$ is an integral curve for the function $f(x) =$ ________. If G is a function whose graph is also an integral curve for f, and if $G(1) = 5$, then $G(x) =$ ________.

5. A slope field for the differential equation

$$\frac{dy}{dx} = \frac{2x}{x^2 - 4}$$

has a line segment with slope ________ through the point $(0, 5)$ and has a line segment with slope ________ through the point $(-4, 1)$.

EXERCISE SET 4.2 Graphing Utility C CAS

1. In each part, confirm that the formula is correct, and state a corresponding integration formula.

(a) $\frac{d}{dx}[\sqrt{1+x^2}] = \frac{x}{\sqrt{1+x^2}}$

(b) $\frac{d}{dx}\left[\frac{1}{3}\sin(1+x^3)\right] = x^2\cos(1+x^3)$

2. In each part, confirm that the stated formula is correct by differentiating.

(a) $\int x\sin x\,dx = \sin x - x\cos x + C$

(b) $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$

FOCUS ON CONCEPTS

3. What is a *constant of integration*? Why does an answer to an integration problem involve a constant of integration?

4. What is an *integral curve* of a function f? How are two integral curves of a function f related?

5–8 Find the derivative and state a corresponding integration formula.

5. $\frac{d}{dx}[\sqrt{x^3+5}]$

6. $\frac{d}{dx}\left[\frac{x}{x^2+3}\right]$

7. $\frac{d}{dx}[\sin(2\sqrt{x})]$

8. $\frac{d}{dx}[\sin x - x\cos x]$

9–10 Evaluate the integral by rewriting the integrand appropriately, if required, and applying the power rule (Formula 2 in Table 4.2.1).

9. (a) $\int x^8\,dx$ (b) $\int x^{5/7}\,dx$ (c) $\int x^3\sqrt{x}\,dx$

(a) $\int \sqrt[3]{x^2}\,dx$ (b) $\int \frac{1}{x^6}\,dx$ (c) $\int x^{-7/8}\,dx$

11–14 Evaluate each integral by applying Theorem 4.2.3 and Formula 2 in Table 4.2.1 appropriately.

11. $\int \left[5x + \frac{2}{3x^5}\right] dx$

12. $\int \left[x^{-1/2} - 3x^{7/5} + \frac{1}{9}\right] dx$

13. $\int [x^{-3} - 3x^{1/4} + 8x^2]\,dx$

$\int \left[\frac{10}{y^{3/4}} - \sqrt[3]{y} + \frac{4}{\sqrt{y}}\right] dy$

15–30 Evaluate the integral and check your answer by differentiating.

15. $\int x(1+x^3)\,dx$

16. $\int (2+y^2)^2\,dy$

17. $\int x^{1/3}(2-x)^2\,dx$

$\int (1+x^2)(2-x)\,dx$

$\int \frac{x^5+2x^2-1}{x^4}\,dx$

20. $\int \frac{1-2t^3}{t^3}\,dt$

21. $\int [3\sin x - 2\sec^2 x]\,dx$

22. $\int [\csc^2 t - \sec t\tan t]\,dt$

23. $\int \sec x(\sec x + \tan x)\,dx$

$\int \csc x(\sin x + \cot x)\,dx$

25. $\int \frac{\sec\theta}{\cos\theta}\,d\theta$

26. $\int \frac{dy}{\csc y}$

$\int \frac{\sin x}{\cos^2 x}\,dx$

28. $\int \left[\phi + \frac{2}{\sin^2\phi}\right] d\phi$

29. $\int [1 + \sin^2\theta\csc\theta]\,d\theta$

$\int \frac{\sec x + \cos x}{2\cos x}\,dx$

Evaluate the integral

$$\int \frac{1}{1+\sin x}\,dx$$

by multiplying the numerator and denominator by an appropriate expression.

32. Use the double-angle formula $\cos 2x = 2\cos^2 x - 1$ to evaluate the integral

$$\int \frac{1}{1+\cos 2x}\,dx$$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

34. If C denotes a constant of integration, the two formulas

$$\int \cos x\,dx = \sin x + C$$

$$\int \cos x\,dx = (\sin x + \pi) + C$$

are both correct equations.

35. The function $f(x) = \sec x + 1$ is a solution to the initial-value problem

$$\frac{dy}{dx} = \sec x\tan x, \quad y(0) = 1$$

36. Every integral curve of the slope field

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

is the graph of an increasing function of x.

37. Use a graphing utility to generate some representative integral curves of the function $f(x) = 5x^4 - \sec^2 x$ over the interval $(-\pi/2, \pi/2)$.

38. Use a graphing utility to generate some representative integral curves of the function $f(x) = (x^2-1)/x^2$ over the interval $(0, 5)$.

39–40 Solve the initial-value problems.

39. (a) $\frac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

(b) $\frac{dy}{dt} = \sin t + 1,\ y\left(\frac{\pi}{3}\right) = \frac{1}{2}$

(c) $\frac{dy}{dx} = \frac{x+1}{\sqrt{x}},\ y(1) = 0$

40. (a) $\dfrac{dy}{dx} = \dfrac{1}{(2x)^3}$, $y(1) = 0$

(b) $\dfrac{dy}{dt} = \sec^2 t - \sin t$, $y\left(\dfrac{\pi}{4}\right) = 1$

(c) $\dfrac{dy}{dx} = x^2\sqrt{x^3}$, $y(0) = 0$

41–44 A particle moves along an s-axis with position function $s = s(t)$ and velocity function $v(t) = s'(t)$. Use the given information to find $s(t)$. ■

41. $v(t) = 32t$; $s(0) = 20$ **42.** $v(t) = \cos t$; $s(0) = 2$

43. $v(t) = 3\sqrt{t}$; $s(4) = 1$ **44.** $v(t) = \sin t$; $s(0) = 0$

45. Find the general form of a function whose second derivative is $\sqrt{x}$. [*Hint:* Solve the equation $f''(x) = \sqrt{x}$ for $f(x)$ by integrating both sides twice.]

46. Find a function f such that $f''(x) = x + \cos x$ and such that $f(0) = 1$ and $f'(0) = 2$. [*Hint:* Integrate both sides of the equation twice.]

47–51 Find an equation of the curve that satisfies the given conditions. ■

47. At each point (x, y) on the curve the slope is $2x + 1$; the curve passes through the point $(-3, 0)$.

48. At each point (x, y) on the curve the slope is $(x + 1)^2$; the curve passes through the point $(-2, 8)$.

49. At each point (x, y) on the curve the slope is $-\sin x$; the curve passes through the point $(0, 2)$.

50. At each point (x, y) on the curve the slope equals the square of the distance between the point and the y-axis; the point $(-1, 2)$ is on the curve.

51. At each point (x, y) on the curve, y satisfies the condition $d^2y/dx^2 = 6x$; the line $y = 5 - 3x$ is tangent to the curve at the point where $x = 1$.

C **52.** In each part, use a CAS to solve the initial-value problem.

(a) $\dfrac{dy}{dx} = x^2 \cos 3x$, $y(\pi/2) = -1$

(b) $\dfrac{dy}{dx} = \dfrac{x^3}{(4 + x^2)^{3/2}}$, $y(0) = -2$

53. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = x$ in the region $-5 \le x \le 5$ and $-5 \le y \le 5$.

(b) Graph some representative integral curves of the function $f(x) = x$.

(c) Find an equation for the integral curve that passes through the point $(2, 1)$.

54. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = \sqrt{x}$ in the region $0 \le x \le 10$ and $-5 \le y \le 5$.

(b) Graph some representative integral curves of the function $f(x) = \sqrt{x}$ for $x > 0$.

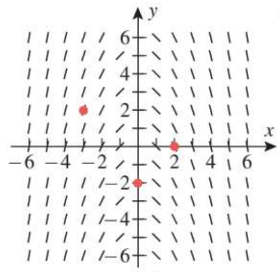
(c) Find an equation for the integral curve that passes through the point $(0, 1)$.

55–58 The given slope field figure corresponds to one of the differential equations below. Identify the differential equation that matches the figure, and sketch solution curves through the highlighted points.

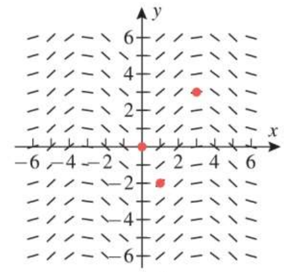
(a) $\dfrac{dy}{dx} = 2$ (b) $\dfrac{dy}{dx} = -x$

(c) $\dfrac{dy}{dx} = x^2 - 4$ (d) $\dfrac{dy}{dx} = \sin x$ ■

55.



56.



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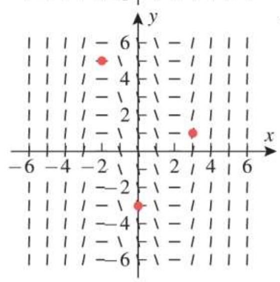
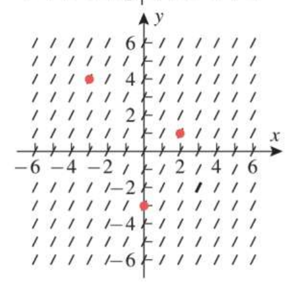


58.



FOCUS ON CONCEPTS

59. Let F and G be the functions defined by

$$F(x) = \frac{x \sin x}{x} \quad \text{and} \quad G(x) = \begin{cases} 2 + \sin x, & x > 0 \\ -1 + \sin x, & x < 0 \end{cases}$$

(a) Show that F and G have the same derivative.

(b) Show that $G(x) \neq F(x) + C$ for any constant C.

(c) Do parts (a) and (b) contradict Theorem 4.2.2? Explain.

60. Follow the directions of Exercise 59 using

$$F(x) = \frac{x^2 + 3x}{x} \quad \text{and} \quad G(x) = \begin{cases} x + 3, & x > 0 \\ x, & x < 0 \end{cases}$$

61–62 Use a trigonometric identity to evaluate the integral. ■

61. $\displaystyle\int \tan^2 x\, dx$ **62.** $\displaystyle\int \cot^2 x\, dx$

63–64 Use the identities $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$ to help evaluate the integrals ■

63. $\displaystyle\int \sin^2(x/2)\, dx$ **64.** $\displaystyle\int \cos^2(x/2)\, dx$

65. The speed of sound in air at 0°C (or 273 K on the Kelvin scale) is 1087 ft/s, but the speed v increases as the temperature T rises. Experimentation has shown that the rate of change of v with respect to T is

$$\frac{dv}{dT} = \frac{1087}{2\sqrt{273}} T^{-1/2}$$

where v is in feet per second and T is in kelvins (K). Find a formula that expresses v as a function of T.

66. The time t between tosses of a juggling ball is a function of the height h of the toss. Suppose that a ball tossed 4 feet high spends 1 second in the air and that the rate of change of t with respect to h is

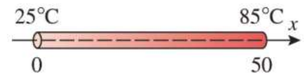
$$\frac{dt}{dh} = \frac{1}{4\sqrt{h}}$$

Find a formula that expresses t as a function of h.

Suppose that a uniform metal rod 50 cm long is insulated laterally, and the temperatures at the exposed ends are maintained at 25°C and 85°C, respectively. Assume that an x-axis is chosen as in the accompanying figure and that the temperature $T(x)$ satisfies the equation

$$\frac{d^2T}{dx^2} = 0$$

Find $T(x)$ for $0 \leq x \leq 50$.



◄ Figure Ex-67

Writing What is an *initial-value problem*? Describe the sequence of steps for solving an initial-value problem.

69. **Writing** What is a *slope field*? How are slope fields and integral curves related?

✓ **QUICK CHECK ANSWERS 4.2** 1. $F'(x) = f(x)$ 2. (a) $\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$ (b) $\int \cos x\,dx = \sin x + C$ 3. (a) $\frac{1}{4}x^4 + \frac{1}{2}x^2 + 5x + C$ (b) $\tan x + \csc x + C$ 4. $2x + 1$; $x^2 + x + 3$ 5. 0; $-\frac{2}{3}$

4.3 INTEGRATION BY SUBSTITUTION

In this section we will study a technique, called ***substitution****, that can often be used to transform complicated integration problems into simpler ones.*

u-SUBSTITUTION

The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \quad (1)$$

or since F is an antiderivative of f,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \quad (2)$$

For our purposes it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x)\,dx$. With this notation (2) can be expressed as

$$\int f(u)\,du = F(u) + C \quad (3)$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

is called the ***method of u-substitution.*** Here the differential notation serves primarily as a useful "bookkeeping" device for the method of u-substitution. The following example illustrates how the method works.

► **Example 1** Evaluate $\int (x^2 + 1)^{50} \cdot 2x\,dx$.

Solution. If we let $u = x^2 + 1$, then $du/dx = 2x$, which implies that $du = 2x\,dx$. Thus, the given integral can be written as

$$\int (x^2 + 1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2 + 1)^{51}}{51} + C \blacktriangleleft$$

It is important to realize that in the method of u-substitution you have control over the choice of u, but once you make that choice the value of du is *computed.* The method of