2.1 The Tangent and Velocity Problems

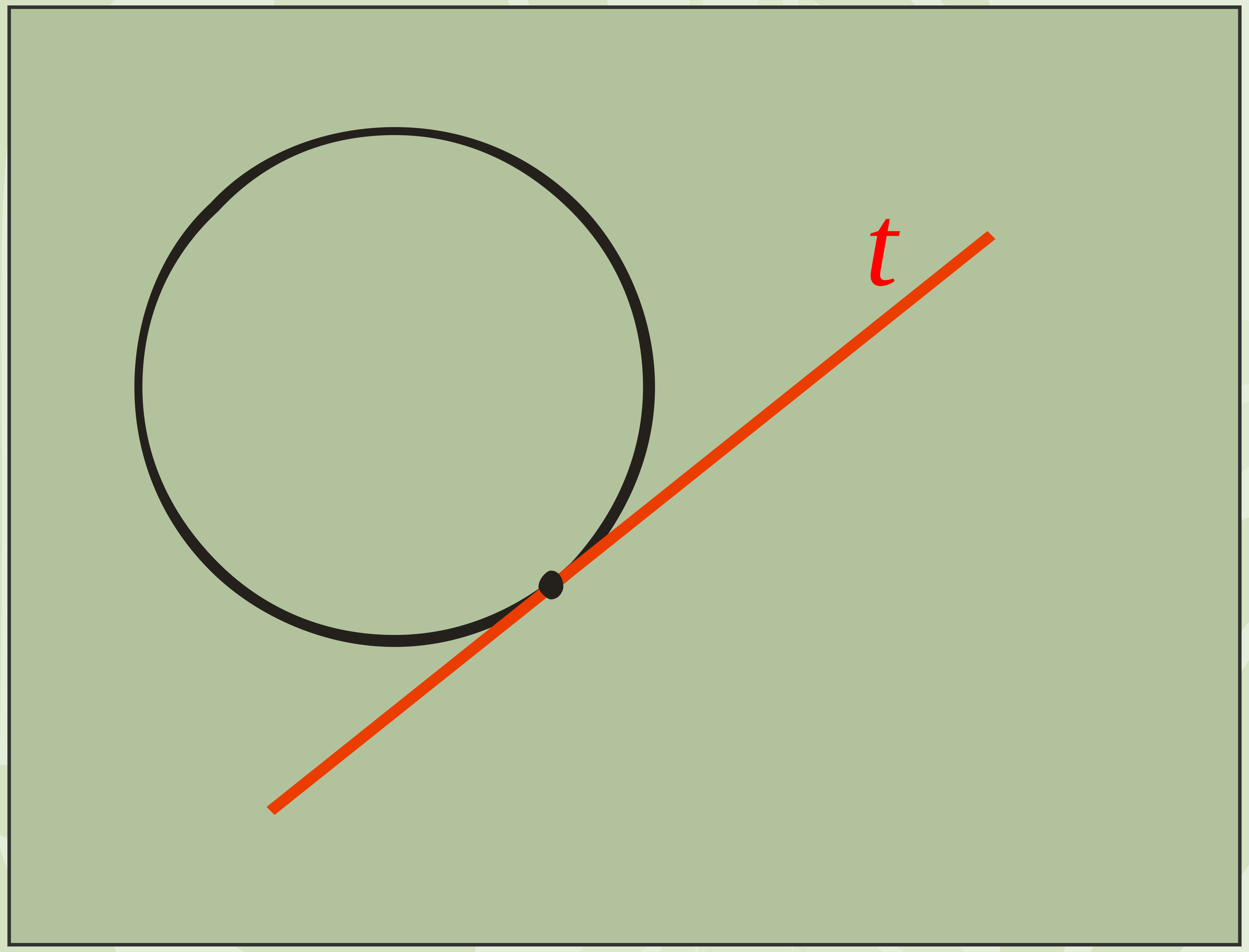
Objective:

To discuss tangent and velocity problems.

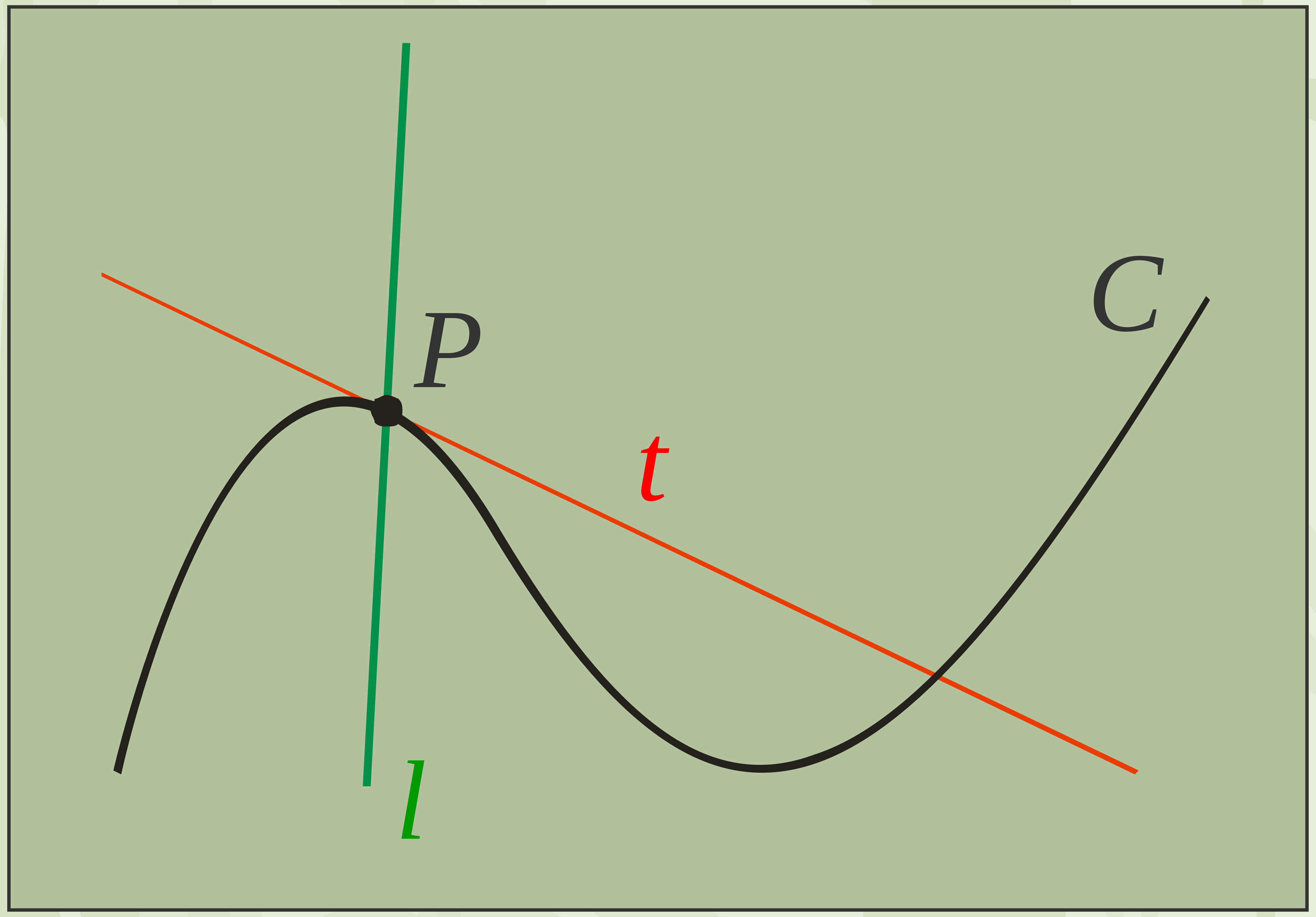
The Tangent Problem

Tangent is from Latin word *tangens*, means "touching"

A tangent to a curve: *a line that touches the curve*



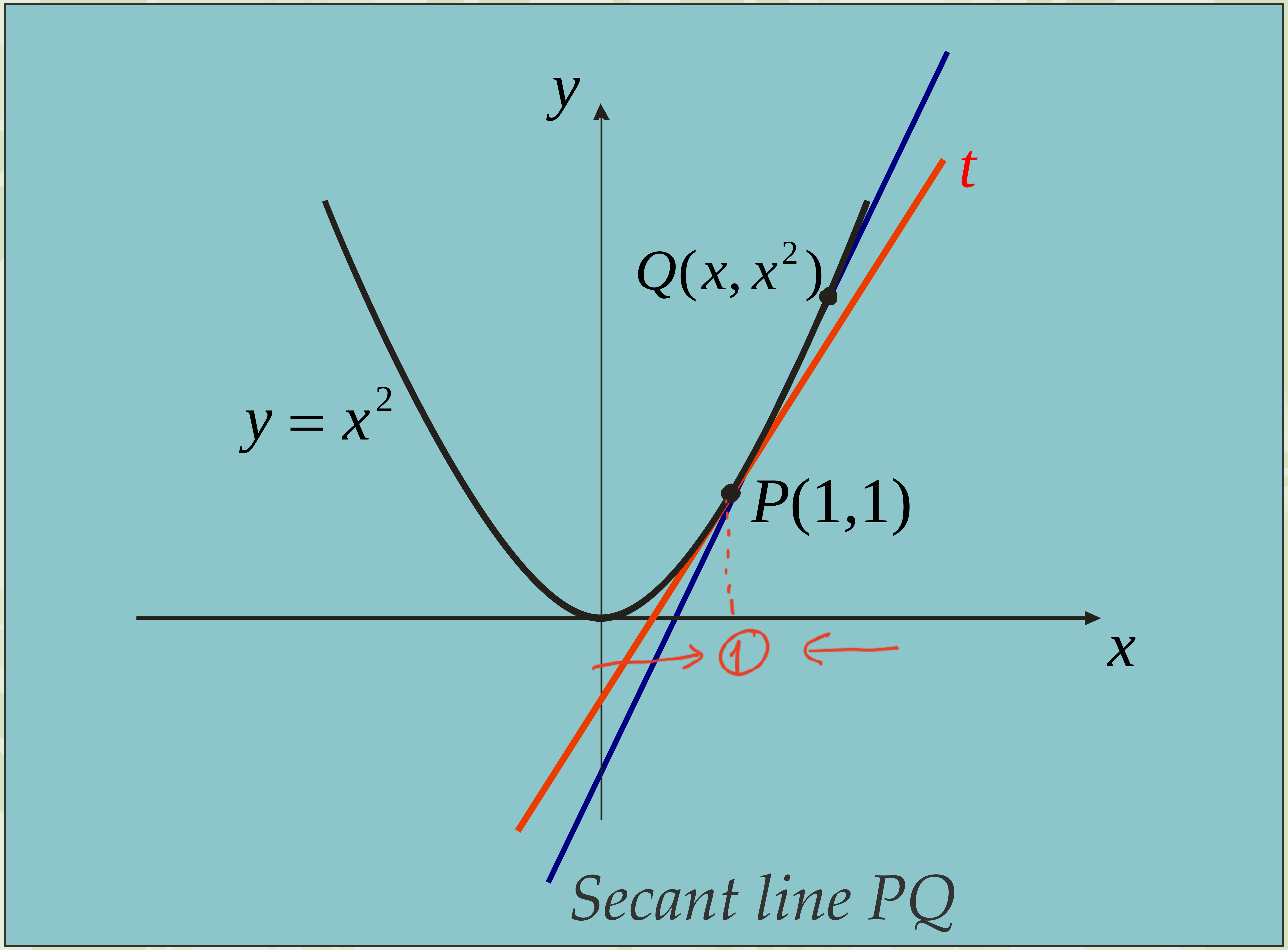
Line *t* is tangent to the circle



Line *t* is tangent to the curve C

The Tangent Problem **Example 1**

Find an equation of tangent line t



equ of straight line

$$y = m x + b$$

$m \rightarrow$ slope of the straight line

$= \dfrac{y_2 - y_1}{x_2 - x_1}$

$b \rightarrow$ y-intercept

sol equation curve $y = x^2$

point $(1, 1)$ P (x_1, y_1)

point $Q \rightarrow (x, x^2)$ (x_2, y_2)

slope of secant PQ $= \dfrac{x^2 - 1}{x - 1}$

$x \neq 1$

As x Approaches 1, slope approaches 2

As $x \to 1$

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

point $P(1, 1)$

equation of Tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 2 + 1$$

$$\boxed{y = 2x - 1}$$ → equation of tangent line

The Tangent Problem **Example 1**

As Q approaches P along the parabola, the corresponding **secant lines** rotate about and approach the tangent line .

The Tangent Problem

Example 1

We choose $x \neq 1$ or $P \neq Q$ and compute the slope of the secant lines:

$$M_{PQ} = \frac{x^2 - 1}{x - 1}$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The Tangent Problem

Example 1

The slope of the tangent line is the limit of the secant line written as

$$m = \lim_{P \to Q} M_{PQ} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

An equation of the tangent line is

$$y - 1 = m(x - 1)$$

That is

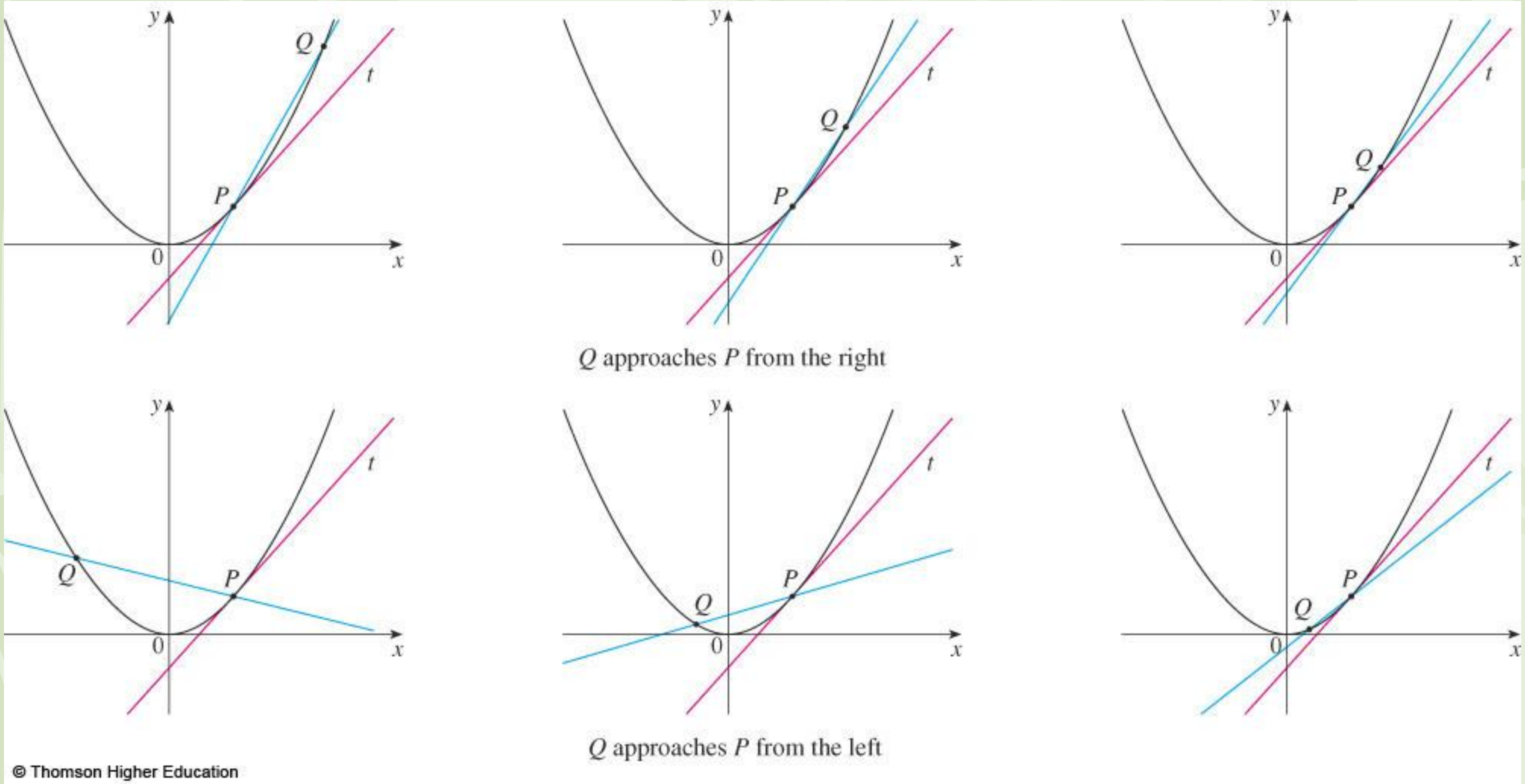
$$y - 1 = 2(x - 1)$$

or

$$y = 2x - 1$$

The Tangent Problem **Example 1**

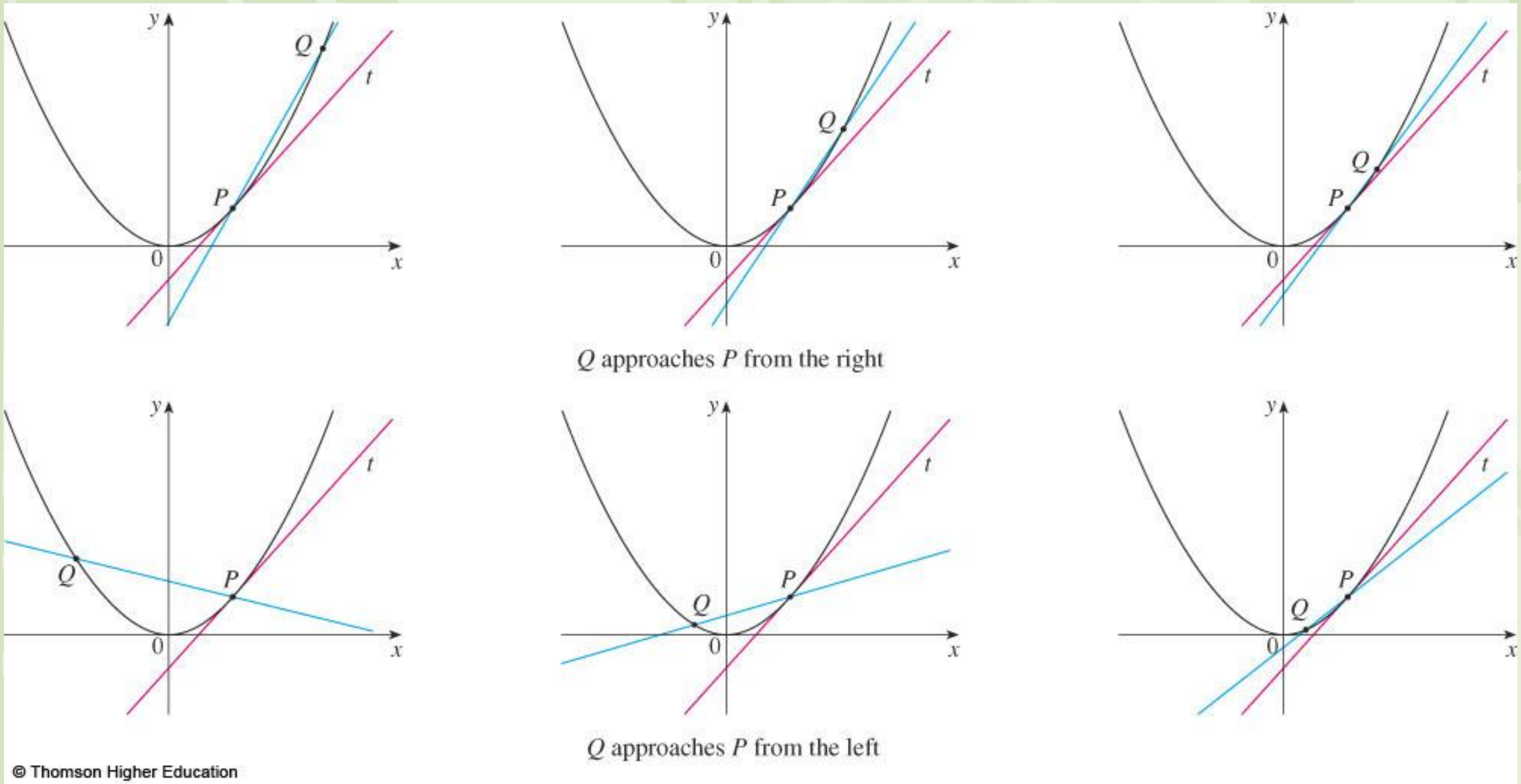
The figure illustrates the limiting process that occurs in this example.



The Tangent Problem

Example 1

As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t.



The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn’t stay still for very long. That is, the velocity of the car is not constant.

- We assume from watching the speedometer that the car has a definite velocity at each moment.
- How is the ‘instantaneous’ velocity defined?

The Velocity Problem

Example 2

Investigate the example of a falling ball.

- Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground.
- Find the velocity of the ball after 5 seconds.

The Velocity Problem **Example 2**

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling.

- Remember, this model neglects air resistance.

The Velocity Problem

Example 2

If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the following equation.

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that you are dealing with a single instant of time ($t = 5$).

- No time interval is involved.

The Velocity Problem **Example 2**

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second (from $t = 5$ to $t = 5.1$).

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1}$$

$$= 49.49 \ \text{m/s}$$

The Velocity Problem **Example 2**

The table shows the results of similar calculations of the average velocity over successively smaller time periods.

- It appears that, as we shorten the time period, the average velocity is becoming closer to 49 m/s.

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

The Velocity Problem **Example 2**

- The instantaneous velocity when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$.
- Thus, the (instantaneous) velocity after 5 s is:

$$v = 49 \text{ m/s}$$

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

The Velocity Problem

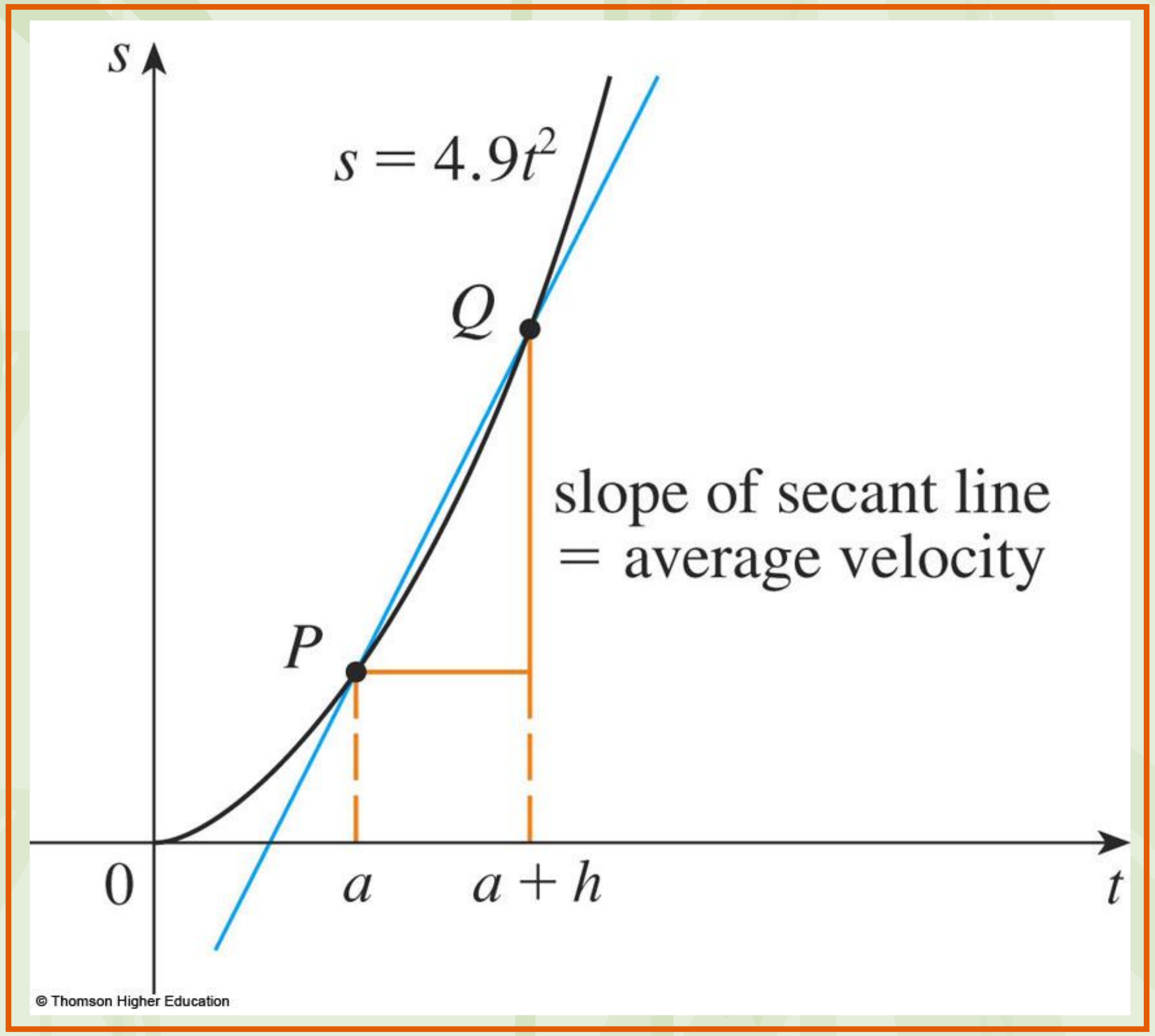
Example 2

You may have the feeling that the calculations used in solving the problem are very similar to those used earlier to find tangents.

- There is a close connection between the tangent problem and the problem of finding velocities.

The Velocity Problem **Example 2**

If we draw the graph of the distance function of the ball and consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$, then the slope of the secant line PQ is:

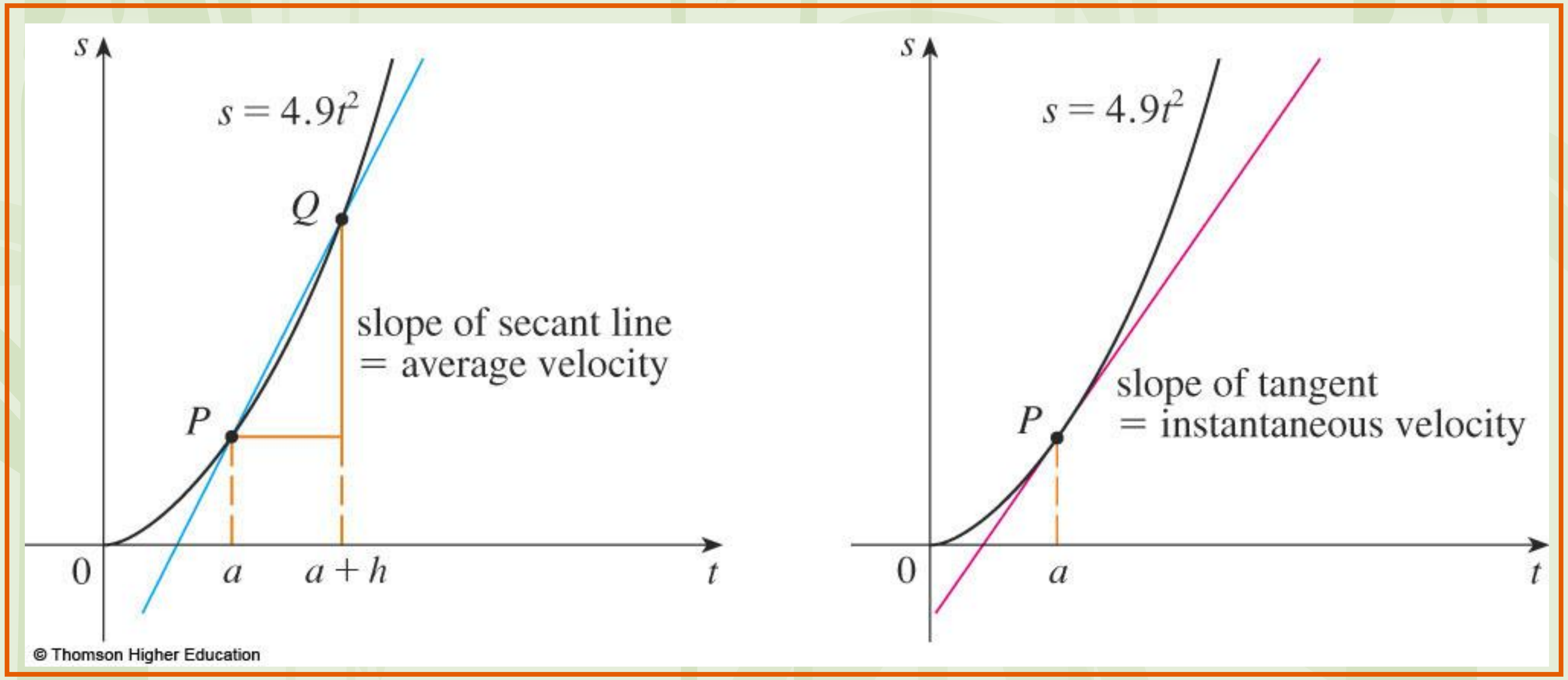


The Velocity Problem

Example 2

That is the same as the average velocity over the time interval $[a, a + h]$.

- Therefore, the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).



The Tangent and Velocity Problems

Examples 1 and 2 show that to solve tangent and velocity problems we must be able to find limits.

After studying methods for computing limits for the next five sections we will return to the problem of finding tangents and velocities in Section 2.7.