SYLLABUS

Part I: COURSE INFORMATION

Semester	471	Academic Year	2025-2026
Department	General Studies	Specialization	Mathematics
Course Title	Differential Equations and Linear Algebra		
Course Code	MATH 222	Prerequisite	MATH 211
Credit Hours	3	Weekly Contact Hours	3

Part II: TEXTBOOK

1.	Edwards, C.H., Penney, D.E., Calvis, D.T. (2018). Differential Equations & Linear Algebra (4th ed.). Pearson. ISBN13: 978-0-13-449718-1

Part III: COURSE DESCRIPTION

This course is designed to provide fundamentals of ordinary differential equations and linear algebra with some useful applications needed for technical courses. It includes topics on first-order differential equations and mathematical models, integral solutions, separable equations, linear equations, other solution methods and exact equations, a review of matrix inverses and determinants, vector spaces and their subspaces, independence of vectors, bases and dimensions, row and column spaces, higher-order differential equations, homogeneous and nonhomogeneous equations, the eigenvalue problem and their applications on diagonalization and powers of matrices, matrices and systems of linear differential equations, the eigenvalue method for linear systems, multiple eigenvalue solutions, matrix exponentials, and nonhomogeneous linear systems.

Part IV: COURSE LEARNING OUTCOME (CLO) – ASSESSMENT CHART

Upon completion of the course, the student should be able to:	Assessment *							
	Q1	Q2	HW1	ME	Q3	Q4	HW2	FE
CLO 2.1. analyze concepts and procedures related to linear algebra and ordinary differential equations			5	2			5	4
CLO 2.2. solve first-order differential equations and related applications	7.5			9.5				5
CLO 2.3. apply methods of linear systems and vector spaces to solve associated algebraic and geometric problems		7.5		8.5				4
CLO 2.4. solve higher-order linear differential equations and related applications					5			9
CLO 2.5. apply the eigenvalue method to solve problems involving diagonalization and powers of matrices					2.5			5.5
CLO 2.6. solve systems of linear differential equations and their applications using the eigenvalue method						7.5		12.5
TOTAL	**7.5**	**7.5**	**5**	**20**	**7.5**	**7.5**	**5**	**40**

* Q = Quiz HW = Homework ME = Midterm Exam FE = Final Exam (comprehensive)

Part V: ATTENDANCE

1. Check your EDUGATE account for the official class time, location, and absences.
2. Attendance will be checked at the start of each period as per ETA Rules and Regulations:
   a. If you do not attend the class for any reason, you will be marked ABSENT for the corresponding missed periods.
   b. If you arrive after the first 5 minutes, you will be marked ABSENT for that period.
   c. If you arrive after the instructor called all students' names/IDs but during the first 5 minutes, you will be marked LATE for that period and will count as 1/3 of an absence.
   d. If you were late three times, then these will be counted as 1 full absence.
3. In a semester, you will get a "**DN**" status if you exceed **15% absences**.
4. If you have a valid reason, and wanted your absence to be removed, contact the Office of Student Affairs and present your valid excuse there.

Part VI: TIPS ON EFFECTIVE STUDYING

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
CLASS	Study Time		CLASS	Study Time		Study Time

- Schedule your study time and be consistent over the semester.
- Copy lecture notes and ask questions. — copy notes during the class and clarify your doubts
- Apply the Feynman technique for learning new concepts: — can be used before class or after class
  1. **Choose** a concept and organize it in your own words. — organize a concept using simple language and write it on a paper
  2. **Teach** the concept to a beginner. — explain the concept to a beginner; identify weaknesses and issues in your explanation
  3. **Fill the Gaps** that you may have missed. — in case you have trouble explaining, get feedback from your audience and research the concept again – go back to your resources or find new resources to fill the gaps
  4. **Refine** your notes and explanation of the concept. — further simplify and refine your notes and explanations; repeat these steps as needed
- Test yourself on the topics of the course. — solve the practice exercises and previous tests or exams
- Do not simply memorize. Studying is not memorizing. — you MUST apply your knowledge and develop mathematical skills in this course

Part VII: Examples, Exercises, Homework, Office Hours

1. The suggested examples and practice exercises are the MINIMUM required level of mathematical maturity to be successful in this course.
2. Answer all the practice exercises without looking at any course-related materials.
3. Do your homework seriously, learn from it, and use it to extend your knowledge and skills in the course.
4. Utilize your instructor's office hours for help, as needed.

Part VIII: PACING SCHEDULE & PRACTICE EXERCISES				
Week No.	**Textbook's Section (§)**	**TOPIC**	**SUGGESTED EXAMPLES**	**REQUIRED PRACTICE EXERCISES**
1-4	1.1	Differential Equations and Mathematical Models	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	3, 6, 7, 10, 12, 16, 19, 26
	1.2	Integrals as General and Particular Solutions	1, 2, 3, 4	4, 7, 10, 14, 18, 19, 25, 31, 35, 42
	1.4	Separable Equations and Applications	1, 2, 3, 4, 5, 6	8, 13, 16, 18, 27, 33, 35, 49, 44, 50, 51
	1.5	Linear First-Oder Equations	1, 2, 3, 4	11, 14, 18, 21, 25, 27, 33
	1.6	Substitution Methods and Exact Equations	1, 2, 3, 4, 5, 6, 8, 9, 10, 11	7, 12, 16, 22, 27, 34, 37
	3.5	Inverses of Matrices (*Review*)	1, 2, 4, 5, 6, 7, 8	3, 12, 15, 18, 21, 25
	3.6	Determinants (*Review*)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	3, 6, 7, 11, 15, 29, 32, 36
	Quiz 1	Assessment for CLO 2.2		
5-7	4.1	The Vector Space $\mathbb{R}^3$	1, 2, 3, 4, 5, 6	3, 8, 14, 17, 23, 28, 33, 36
	4.2	The Vector Space $\mathbb{R}^n$ and Subspaces	1, 2, 3, 4, 5	3, 11, 15, 18, 19, 22
	4.3	Linear Combinations and Independence of Vectors	1, 2, 3, 4, 5, 6	3, 8, 10, 16, 19, 25
	4.4	Bases and Dimension for Vector Spaces	1, 2, 3, 4, 5, 6, 7, 8, 9, 11	5 ,9, 15, 18, 26, 29
	4.5	Row and Column Spaces	1, 2, 3, 4, 5	2, 11, 14, 19, 21
	Quiz 2	Assessment for CLO 2.3		
8	**Midterm Exam**	Assessment for CLO 2.1 (10.00%), CLO 2.2 (47.50%), and CLO 2.3 (42.50%)		

Part VIII: PACING SCHEDULE & PRACTICE EXERCISES				
Week No.	**Textbook's Section (§)**	**TOPIC**	**SUGGESTED EXAMPLES**	**REQUIRED PRACTICE EXERCISES**
	5.1	Second-Order Linear Equations	1, 2, 3, 4, 5, 6, 7	3, 12, 17, 20, 33, 41, 46, 53
	5.2	General Solutions of Linear Equations	1, 2, 3, 4, 5, 6, 7	5, 9, 11, 12, 15, 21, 27, 30
Q3	5.3	Homogeneous Equations with Constant Coefficients	1, 2, 3, 4, 5, 6, 7, 8	4, 10, 13, 20, 22, 25, 29, 32, 35, 40
	5.5	Nonhomogeneous Equations and Undetermined Coefficients	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	3, 9, 23, 26, 33
9-12	6.1	Introduction to Eigenvalues	1, 2, 3, 4, 5, 6, 7	1, 8, 13, 18, 26, 30, 31
	6.2	Diagonalization of Matrices	1, 2, 3, 4	2, 9, 13, 19, 23, 27
	6.3	Applications Involving Powers of Matrices	1, 2, 3, 4, 5, 6	1, 9, 14, 20, 29, 34, 36
	7.1	First-Order Systems and Applications	1, 2, 3, 4, 5, 6, 7	2, 4, 9, 13, 19, 25
	Quiz 3	Assessment for CLO 2.4 (66.67%) and CLO 2.5 (33.33%)		
	7.2	Matrices and Linear Systems	1, 2, 3, 4, 5	1, 5, 6, 13, 18, 20, 24, 30, 33
Q4	7.3	The Eigenvalue Method for Linear Systems	1, 2, 3, 4	2, 3, 9, 15, 17, 23, 26
	7.5	Second-Order Systems and Mechanical Applications	1, 2, 3	3, 5, 7, 9
13-15	7.6	Multiple Eigenvalue Solutions	1, 2, 3, 4, 5, 6, 7, 8	1, 7, 11, 15, 21, 25
	8.1	Matrix Exponentials and Linear Systems	1, 2, 3, 4, 5, 6, 7	1, 7, 12, 17, 21, 24, 25, 27
	8.2	Nonhomogeneous Linear Systems	1, 2, 3, 4	1, 5, 8, 13, 20, 22, 27, 30
	Quiz 4	Assessment for CLO 2.6		
16-17	**Final Exam**	Assessment for CLO 2.1 (10.00%), CLO 2.2 (12.50%), CLO 2.3 (10.00%), CLO 2.4 (22.50%), CLO 2.5 (13.75%), and CLO 2.6 (31.25%)		

Differential Equations & Linear Algebra

FOURTH EDITION

C. HENRY EDWARDS
DAVID E. PENNEY
DAVID T. CALVIS

Pearson

DIFFERENTIAL EQUATIONS & LINEAR ALGEBRA

Fourth Edition

C. Henry Edwards

David E. Penney

The University of Georgia

David T. Calvis

Baldwin Wallace University

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CONTENTS

Review

1 First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Example 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

1st

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

2nd

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we

→ First order

Second order

Third order

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3rd order

$$\Rightarrow y''' + 7x\,y'' + y' = 0$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} + 6\frac{\partial' y}{\partial x'} - 5y = e^x$$

2nd order

Diff Eq $\longrightarrow$ معادلة تفاضلية

↳ $y' + y'' + \ldots = 0$

↳ $\dfrac{d^2y}{dx^2} \quad \dfrac{dy}{dx}$

↳ $\dddot{y} \quad \ddot{y} + \ldots$

$$\boxed{\begin{array}{l} y^{(1)} + 5y = 7x \\ \hookrightarrow \text{1st order DE} \end{array}}$$

are challenged to find the unknown *functions* $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$—that is, the differential equation

$$\frac{dy}{dx} = 2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation, if possible.

Example 2 If C is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant C. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1). ■

Differential Equations and Mathematical Models

The following three examples illustrate the process of translating scientific laws and principles into differential equations. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

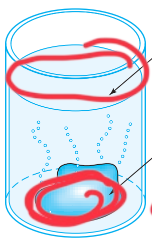

FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

Example 3 **Rate of cooling** Newton's law of cooling may be stated in this way: The *time rate of change* (the rate of change with respect to time t) of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium (Fig. 1.1.1). That is,

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$$\frac{dT}{dt} = -k(T - A), \tag{3}$$

where k is a positive constant. Observe that if $T > A$, then $dT/dt < 0$, so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then $dT/dt > 0$, so that T is increasing.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body. ■

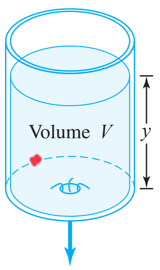

FIGURE 1.1.2. Torricelli's law of draining, Eq. (4), describes the draining of a water tank.

Example 4 **Draining tank** Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank (Fig. 1.1.2) is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt} = -k\sqrt{y}, \tag{4}$$

where k is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$, so $dV/dt = A \cdot (dy/dt)$. In this case Eq. (4) takes the form

$$\frac{dy}{dt} = -h\sqrt{y}, \tag{5}$$

where $h = k/A$ is a constant. ■

$$\frac{\partial y}{\partial x} = 2xy \rightarrow \text{1st order DE}$$

Ex ② $\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} C e^{x^2}$

$$\frac{\partial y}{\partial x} = y' = C \left[2x \cdot e^{x^2} \right]$$

$$y = C e^{x^2}$$

$$\boxed{\frac{\partial y}{\partial x} = 2xy}$$

$$\frac{dy}{dx} = 2xy$$

$\hookrightarrow$ $\boxed{y = \underline{c}\, e^{x^2}}$

$\hookrightarrow$ constant

5, -3, . . .

Example 5 **Population growth** The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{6}$$

where k is the constant of proportionality. ■

Let us discuss Example 5 further. Note first that each function of the form

$$\boxed{P(t) = Ce^{kt}} \tag{7}$$

is a solution of the differential equation

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$$\frac{dP}{dt} = kP$$

in (6). We verify this assertion as follows:

$$P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP(t)$$

for all real numbers t. Because substitution of each function of the form given in (7) into Eq. (6) produces an identity, all such functions are solutions of Eq. (6).

Thus, even if the value of the constant k is known, the differential equation $dP/dt = kP$ has *infinitely many* different solutions of the form $P(t) = Ce^{kt}$, one for each choice of the "arbitrary" constant C. This is typical of differential equations. It is also fortunate, because it may allow us to use additional information to select from among all these solutions a particular one that fits the situation under study.

Example 6 **Population growth** Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at time $t = 0$ (hours, h) was 1000, and that the population doubled after 1 h. This additional information about $P(t)$ yields the following equations:

$$1000 = P(0) = Ce^0 = C,$$
$$2000 = P(1) = Ce^k.$$

It follows that $C = 1000$ and that $e^k = 2$, so $k = \ln 2 \approx 0.693147$. With this value of k the differential equation in (6) is

$$\frac{dP}{dt} = (\ln 2)P \approx (0.693147)P.$$

Substitution of $k = \ln 2$ and $C = 1000$ in Eq. (7) yields the particular solution

$$P(t) = 1000e^{(\ln 2)t} = 1000(e^{\ln 2})^t = 1000 \cdot 2^t \qquad (\text{because } e^{\ln 2} = 2)$$

that satisfies the given conditions. We can use this particular solution to predict future populations of the bacteria colony. For instance, the predicted number of bacteria in the population after one and a half hours (when $t = 1.5$) is

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828.$$ ■

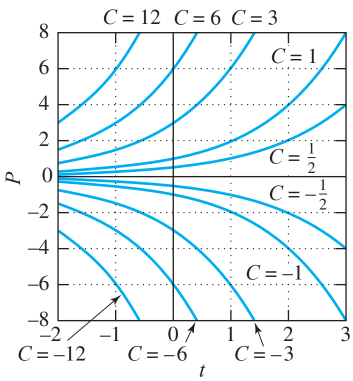

FIGURE 1.1.3. Graphs of $P(t) = Ce^{kt}$ with $k = \ln 2$.

The condition $P(0) = 1000$ in Example 6 is called an **initial condition** because we frequently write differential equations for which $t = 0$ is the "starting time." Figure 1.1.3 shows several different graphs of the form $P(t) = Ce^{kt}$ with $k = \ln 2$. The graphs of all the infinitely many solutions of $dP/dt = kP$ in fact fill the entire two-dimensional plane, and no two intersect. Moreover, the selection of any one point P_0 on the P-axis amounts to a determination of $P(0)$. Because exactly one solution passes through each such point, we see in this case that an initial condition $P(0) = P_0$ determines a unique solution agreeing with the given data.

Ex 6

$$\frac{\partial P}{\partial t} = kP$$

$$P(t) = C e^{kt} \qquad \boxed{e^0 = 1}$$

$t = 0 \longrightarrow P = 1000$

$t = 1 \longrightarrow P = 2 \times 1000 = 2000$

$t = 0 \longrightarrow P = \boxed{1000 = C}\ e^{0}$!

$C = 1000$ ✓

$$\boxed{P(t) = 1000\, e^{kt}}$$

$t = 1$

$$\frac{\cancel{2000}}{\cancel{1000}} = \frac{\cancel{1000}\, e^{k}}{\cancel{1000}}$$

$$\ln 2 = \cancel{\ln} \cancel{e}^{k}$$

$$\boxed{\ln 2 = k}$$

$$p(t) = 1000\, e^{\ln 2 \cdot t} \rightarrow \left(e^{\ln 2}\right)^{t}$$

$$p(t) = 1000 \left(\cancel{e}^{\cancel{\ln} 2}\right)^{t}$$

$$\boxed{p(t) = 1000 \cdot 2^{t}}$$

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

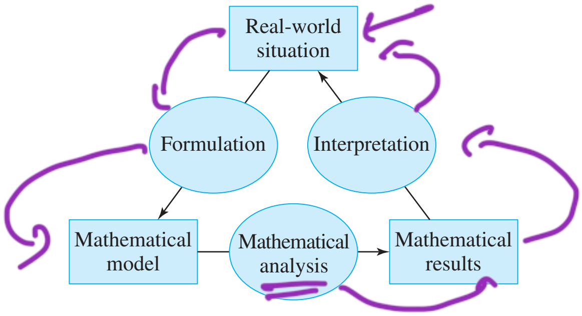

FIGURE 1.1.4. The process of mathematical modeling.

In the population example, the real-world problem is that of determining the population at some future time. A **mathematical model** consists of a list of variables (P and t) that describe the given situation, together with one or more equations relating these variables ($dP/dt = kP$, $P(0) = P_0$) that are known or are assumed to hold. The mathematical analysis consists of solving these equations (here, for P as a function of t). Finally, we apply these mathematical results to attempt to answer the original real-world question.

As an example of this process, think of first formulating the mathematical model consisting of the equations $dP/dt = kP$, $P(0) = 1000$, describing the bacteria population of Example 6. Then our mathematical analysis there consisted of solving for the solution function $P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t$ as our mathematical result. For an interpretation in terms of our real-world situation—the actual bacteria population—we substituted $t = 1.5$ to obtain the predicted population of $P(1.5) \approx 2828$ bacteria after 1.5 hours. If, for instance, the bacteria population is growing under ideal conditions of unlimited space and food supply, our prediction may be quite accurate, in which case we conclude that the mathematical model is adequate for studying this particular population.

On the other hand, it may turn out that no solution of the selected differential equation accurately fits the actual population we're studying. For instance, for *no* choice of the constants C and k does the solution $P(t) = Ce^{kt}$ in Eq. (7) accurately describe the actual growth of the human population of the world over the past few centuries. We must conclude that the differential equation $dP/dt = kP$ is inadequate for modeling the world population—which in recent decades has "leveled off" as compared with the steeply climbing graphs in the upper half ($P > 0$) of Fig. 1.1.3. With sufficient insight, we might formulate a new mathematical model including a perhaps more complicated differential equation, one that takes into account such factors as a limited food supply and the effect of increased population on birth and death rates. With the formulation of this new mathematical model, we may attempt to traverse once again the diagram of Fig. 1.1.4 in a counterclockwise manner. If we can solve the new differential equation, we get new solution functions to compare with the real-world population. Indeed, a successful population analysis may require refining the mathematical model still further as it is repeatedly measured against real-world experience.

But in Example 6 we simply ignored any complicating factors that might affect our bacteria population. This made the mathematical analysis quite simple, perhaps unrealistically so. A satisfactory mathematical model is subject to two contradictory requirements: It must be sufficiently detailed to represent the real-world situation with relative accuracy, yet it must be sufficiently simple to make the mathematical analysis practical. If the model is so detailed that it fully represents the physical situation, then the mathematical analysis may be too difficult to carry out. If the model is too simple, the results may be so inaccurate as to be useless. Thus there is an inevitable tradeoff between what is physically realistic and what is mathematically possible. The construction of a model that adequately bridges this gap between realism and feasibility is therefore the most crucial and delicate step in the process. Ways must be found to simplify the model mathematically without sacrificing essential features of the real-world situation.

Mathematical models are discussed throughout this book. The remainder of this introductory section is devoted to simple examples and to standard terminology used in discussing differential equations and their solutions.

Examples and Terminology

Example 7 If C is a constant and $y(x) = 1/(C - x)$, then

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2$$

if $x \neq C$. Thus

$$y(x) = \frac{1}{C-x} \tag{8}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \tag{9}$$

on any interval of real numbers not containing the point $x = C$. Actually, Eq. (8) defines a *one-parameter family* of solutions of $dy/dx = y^2$, one for each value of the arbitrary constant or "parameter" C. With $C = 1$ we get the particular solution

$$y(x) = \frac{1}{1-x}$$

that satisfies the initial condition $y(0) = 1$. As indicated in Fig. 1.1.5, this solution is continuous on the interval $(-\infty, 1)$ but has a vertical asymptote at $x = 1$. ■

Example 8 Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln x$ satisfies the differential equation

$$4x^2y'' + y = 0 \tag{10}$$

for all $x > 0$.

Solution First we compute the derivatives

$$y'(x) = -\tfrac{1}{2}x^{-1/2}\ln x \quad \text{and} \quad y''(x) = \tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

$$4x^2y'' + y = 4x^2\left(\tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln x = 0$$

if x is positive, so the differential equation is satisfied for all $x > 0$. ■

Ex 7

$$\frac{\partial y}{\partial x} = \frac{d}{dx}\left(\frac{1}{c-x}\right)$$

$$\frac{\partial y}{\partial x} = \frac{0-(-1)}{(c-x)^2} = \frac{+1}{(c-x)^2}$$

$+$

$$\frac{\partial y}{\partial x} = \left(\frac{1}{c-x}\right)^{2}$$

$$y = \frac{1}{c-x}$$

$$\boxed{\frac{\partial y}{\partial x} = y^2}$$

$\rightarrow$ 1st order DE

Ex $\boxed{8}$ $y = 2x^{\frac{1}{2}} - x^{\frac{1}{2}} \cdot \ln x$

$\frac{1}{2} - 1 \qquad \frac{1}{2} - 1$

$4x^2 y'' + y = 0$

$$y' = \cancel{2} \cdot \frac{1}{\cancel{2}} x^{-\frac{1}{2}} - \left[\frac{1}{2} x^{-\frac{1}{2}} \cdot \ln x + x^{\frac{1}{2}} \cdot \frac{1}{x} \right]$$

-1

$$y' = \cancel{x^{-\frac{1}{2}}} - \frac{1}{2} x^{-\frac{1}{2}} \ln x - \cancel{x^{-\frac{1}{2}}}$$

$$\boxed{y' = -\frac{1}{2} x^{-\frac{1}{2}} \ln x}$$

$-\frac{1}{2} - 1$

$$x^{-1}$$

$$y'' = -\frac{1}{2}\left[-\frac{1}{2}x^{-\frac{3}{2}}\ln x + x^{-\frac{1}{2}}\cdot\frac{1}{x}\right]$$

$$\boxed{y'' = +\frac{1}{4}x^{-\frac{3}{2}}\ln x - \frac{1}{2}x^{-\frac{3}{2}}}$$

$$4x^{2}\left[\frac{1}{4}x^{-\frac{3}{2}}\ln x - \frac{1}{2}x^{-\frac{3}{2}}\right]$$

$$+1\left[2x^{\frac{1}{2}} - x^{\frac{1}{2}}\ln x\right]$$

$$x^{2+(-\frac{3}{2})} \ln x - 2x^{2+(-\frac{3}{2})}$$

$$+2x^{\frac{1}{2}} - x^{\frac{1}{2}} \ln x$$

$$\cancel{+x^{\frac{1}{2}} \ln x} \; \cancel{- 2x^{\frac{1}{2}}}$$

$$\cancel{+2x^{\frac{1}{2}}} \; \cancel{- x^{\frac{1}{2}} \ln x} = 0$$

$$4x^2 y'' + y = 0 \to \text{2nd}$$

$$\hookrightarrow y = 2x^{\frac{1}{2}} - x^{\frac{1}{2}} \ln x$$

$s^2 \to 2s$

$(-3)^2 \to 9$

$' \quad '' \quad '''$

(u)

The fact that we can write a differential equation is not enough to guarantee that it has a solution. For example, it is clear that the differential equation

$$(y')^2 + y^2 = -1 \tag{11}$$

has *no* (real-valued) solution, because the sum of nonnegative numbers cannot be negative. For a variation on this theme, note that the equation

$$(y')^2 + y^2 = 0 \tag{12}$$

obviously has only the (real-valued) solution $y(x) \equiv 0$. In our previous examples any differential equation having at least one solution indeed had infinitely many.

The **order** of a differential equation is the order of the highest derivative that appears in it. The differential equation of Example 8 is of second order, those in Examples 2 through 7 are first-order equations, and

$$\boxed{y^{(4)}} + x^2 y^{(3)} + x^5 y = \sin x$$

is a fourth-order equation. The most general form of an ***n*th-order** differential equation with independent variable x and unknown function or dependent variable $y = y(x)$ is

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \tag{13}$$

where F is a specific real-valued function of $n + 2$ variables.

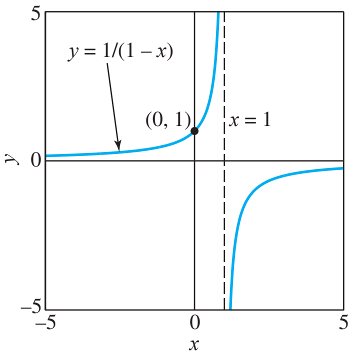

FIGURE 1.1.5. The solution of $y' = y^2$ defined by $y(x) = 1/(1 - x)$.

Our use of the word *solution* has been until now somewhat informal. To be precise, we say that the continuous function $u = u(x)$ is a **solution** of the differential equation in (13) **on the interval** I provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F\left(x, u, u', u'', \dots, u^{(n)}\right) = 0$$

for all x in I. For the sake of brevity, we may say that $u = u(x)$ **satisfies** the differential equation in (13) on I.

Remark Recall from elementary calculus that a differentiable function on an open interval is necessarily continuous there. This is why only a continuous function can qualify as a (differentiable) solution of a differential equation on an interval. ■

$\frac{dy}{dx} \to$ dep

indep

Example 7 **Continued** Figure 1.1.5 shows the two "connected" branches of the graph $y = 1/(1 - x)$. The left-hand branch is the graph of a (continuous) solution of the differential equation $y' = y^2$ that is defined on the interval $(-\infty, 1)$. The right-hand branch is the graph of a *different* solution of the differential equation that is defined (and continuous) on the different interval $(1, \infty)$. So the single formula $y(x) = 1/(1 - x)$ actually defines two different solutions (with different domains of definition) of the same differential equation $y' = y^2$. ■

Example 9 If A and B are constants and

$$y(x) = A\cos 3x + B\sin 3x, \tag{14}$$

then two successive differentiations yield

$$y'(x) = -3A\sin 3x + 3B\cos 3x,$$
$$y''(x) = -9A\cos 3x - 9B\sin 3x = -9y(x)$$

for all x. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$y'' + 9y = 0 \tag{15}$$

on the whole real number line. Figure 1.1.6 shows the graphs of several such solutions. ■

Ex 9 $\quad y'' + 9y = 0$

$y = A\cos 3x + B\sin 3x$

$y' = -3A\sin 3x + 3B\cos 3x$

$$\boxed{y'' = -9A\cos 3x - 9B\sin 3x}$$

$\cancel{-9A\cos 3x} \cancel{-9B\sin 3x}$

$\cancel{+9A\cos 3x} \cancel{+9B\sin 3x} = 0$

φ x z

Although the differential equations in (11) and (12) are exceptions to the general rule, we will see that an nth-order differential equation ordinarily has an n-parameter family of solutions—one involving n different arbitrary constants or parameters.

In both Eqs. (11) and (12), the appearance of y' as an implicitly defined function causes complications. For this reason, we will ordinarily assume that any differential equation under study can be solved explicitly for the highest derivative that appears; that is, that the equation can be written in the so-called *normal form*

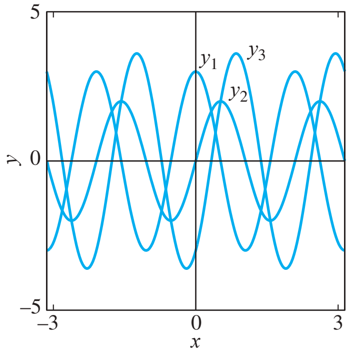

FIGURE 1.1.6. The three solutions $y_1(x) = 3\cos 3x$, $y_2(x) = 2\sin 3x$, and $y_3(x) = -3\cos 3x + 2\sin 3x$ of the differential equation $y'' + 9y = 0$.

$$y^{(n)} = G\left(x, y, y', y'', \dots, y^{(n-1)}\right), \tag{16}$$

where G is a real-valued function of $n+1$ variables. In addition, we will always seek only real-valued solutions unless we warn the reader otherwise.

All the differential equations we have mentioned so far are **ordinary** differential equations, meaning that the unknown function (dependent variable) depends on only a *single* independent variable. If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a **partial** differential equation. For example, the temperature $u = u(x, t)$ of a long thin uniform rod at the point x at time t satisfies (under appropriate simple conditions) the partial differential equation

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$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

$\frac{\partial}{\partial x}$

where k is a constant (called the *thermal diffusivity* of the rod). In Chapters 1 through 8 we will be concerned only with *ordinary* differential equations and will refer to them simply as differential equations.

$\frac{\partial y}{\partial x}$

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In this chapter we concentrate on *first-order* differential equations of the form

$$\frac{dy}{dx} = f(x, y). \tag{17}$$

We also will sample the wide range of applications of such equations. A typical mathematical model of an applied situation will be an **initial value problem**, consisting of a differential equation of the form in (17) together with an **initial condition** $y(x_0) = y_0$. Note that we call $y(x_0) = y_0$ an initial condition whether or not $x_0 = 0$. To **solve** the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{18}$$

means to find a differentiable function $y = y(x)$ that satisfies both conditions in Eq. (18) on some interval containing x_0.

Example 10 Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Solution We need only find a value of C so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

Ex $\dfrac{\partial y}{\partial x} = y^2 \qquad y(1) = 2$

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$$y = \frac{1}{c - x}$$

$$y(1) = 2$$

$x \qquad y$

$$\frac{2}{1} = \frac{1}{(c-1)}$$

$$2c - 2 = 1 + 2$$

$$\frac{\cancel{2}c}{\cancel{2}} = \frac{3}{2} \longrightarrow \boxed{c = \frac{3}{2}}$$

$$y = \frac{1}{\left(\frac{3}{2} - x\right)} * \frac{2}{2}$$

$$\boxed{y = \frac{2}{3 - 2x}}$$

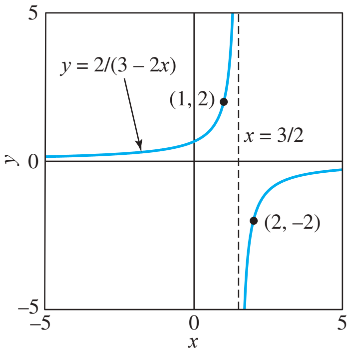

FIGURE 1.1.7. The solutions of $y' = y^2$ defined by $y(x) = 2/(3-2x)$.

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3-2x)$. The left-hand branch is the graph on $(-\infty, \frac{3}{2})$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $(\frac{3}{2}, \infty)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1 + x^2}$
10. $x^2 y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2 y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2 y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x \sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2 y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$. Can you *guess* what the graph of such a function g might look like?
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

Differential Equations as Models

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.
34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.

36. In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. $y'' = 0$

38. $y' = y$

39. $xy' + y = 3x^2$

40. $(y')^2 + y^2 = 1$

41. $y' + y = e^x$

42. $y'' + y = 0$

Problems 43 through 46 concern the differential equation

$$\frac{dx}{dt} = kx^2,$$

where k is a constant.

43. **(a)** If k is a constant, show that a general (one-parameter) solution of the differential equation is given by $x(t) = 1/(C - kt)$, where C is an arbitrary constant.
(b) Determine by inspection a solution of the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** Assume that k is positive, and then sketch graphs of solutions of $x' = kx^2$ with several typical positive values of $x(0)$.
(b) How would these solutions differ if the constant k were negative?

45. Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. Based on the result of Problem 43, how long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

46. Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate of 1 m/s^2 when $v = 5$ m/s. Based on the result of Problem 43, long does it take for the velocity of the boat to decrease to 1 m/s? To $\frac{1}{10}$m/s? When does the boat come to a stop?

47. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. **(a)** Determine a value of C so that $y(10) = 10$. **(b)** Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? **(c)** Figure 1.1.8 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$. Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point (a, b) in the plane, the differential equation $dy/dx = y^2$ has exactly one solution $y(x)$ satisfying the condition $y(a) = b$?

48. **(a)** Show that $y(x) = Cx^4$ defines a one-parameter family of differentiable solutions of the differential equation $xy' = 4y$ (Fig. 1.1.9). **(b)** Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geqq 0 \end{cases}$$

defines a differentiable solution of $xy' = 4y$ for all x, but is not of the form $y(x) = Cx^4$. **(c)** Given any two real numbers a and b, explain why—in contrast to the situation in part (c) of Problem 47—there exist infinitely many differentiable solutions of $xy' = 4y$ that all satisfy the condition $y(a) = b$.

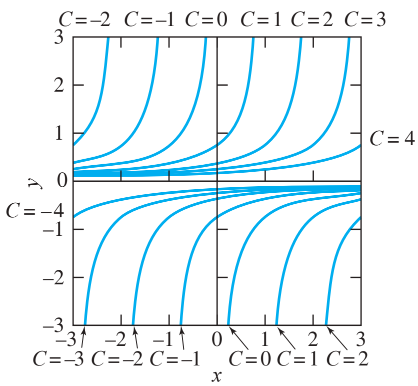

FIGURE 1.1.8. Graphs of solutions of the equation $dy/dx = y^2$.

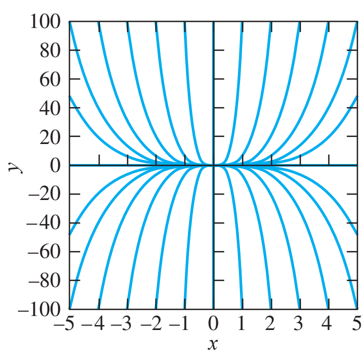

FIGURE 1.1.9. The graph $y = Cx^4$ for various values of C.