ME 441 – CONTROL SYSTEMS

Lecture 2

- Design Of Feedback Control Systems, 4th Edition

R. T. Stefani, C. J. Savant, B. Shahian, G. H. Hostetter

Course Instructor:

Dr. Saad Arif – Assistant Professor – sarif@kfu.edu.sa

Office # 1056, Department of Mechanical Engineering, College of Engineering,

King Faisal University, Al-Ahsa, Saudi Arabia

Lecture Video Links

Lecture-wise: https://youtube.com/playlist?list=PLRsy8EQwnUTPJ_IsvljQzYlGvvAHbZTpU

Topic-wise: https://www.youtube.com/playlist?list=PLRsy8EQwnUTNP5gOU3NtrtyrALE1obZZH

For Further Information, Visit the Channel: www.bit.ly/saadarif

Objectives

- Review of Complex Numbers
- Review of Ordinary Differential Equations
- Review of Laplace Transforms

Complex Numbers ⇒ الآلة حاسبة

- Rectangular Form

$$\boxed{z = x + jy}$$

- Euler Formula

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

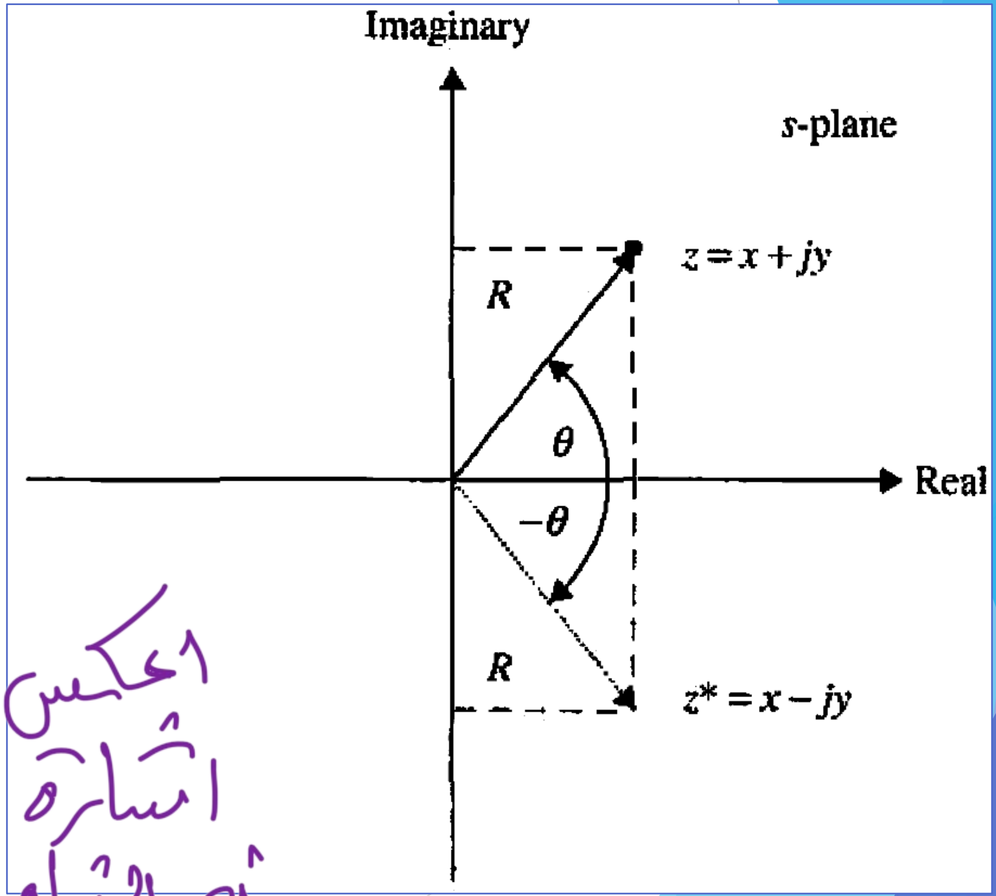
- Polar Form

$$\boxed{z = R e^{j\theta} = R\angle\theta}$$

- Complex Conjugate

$$\boxed{z^* = x - jy}$$

→ اعكس اشارة الزاوية

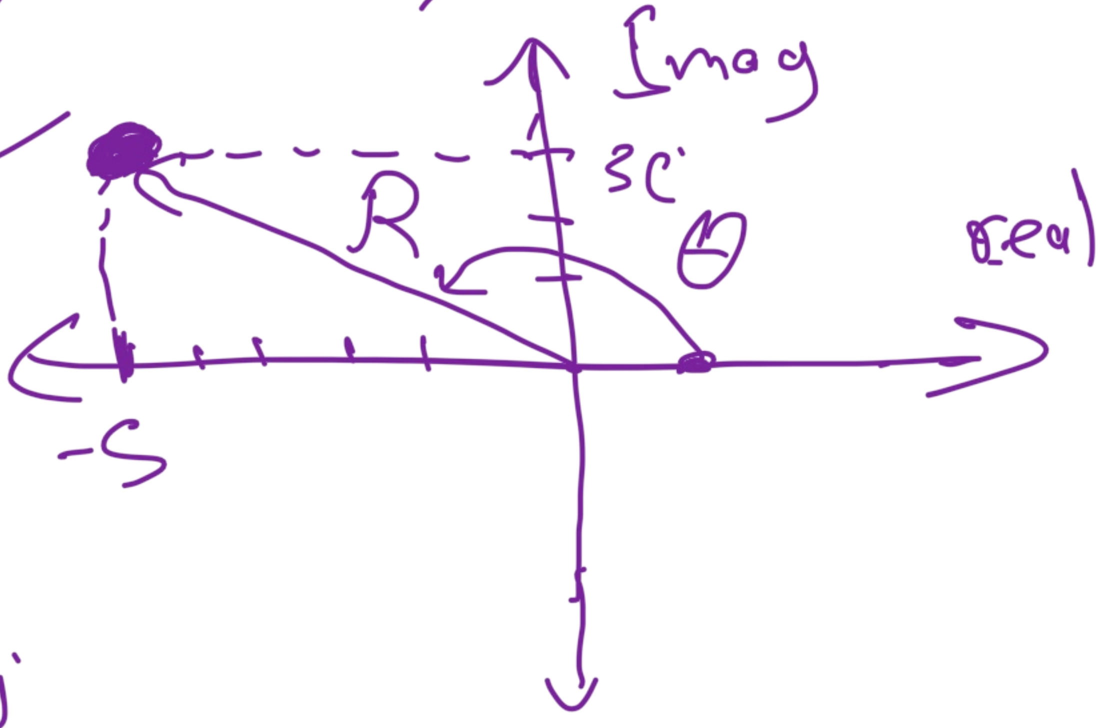


Complex number z

$$\overset{\text{Real}}{-5} + \overset{\text{imag}}{3j}$$

$$z = x + jy \longrightarrow -5 + 3j$$

$$i = j = \sqrt{-1}$$ ✓



Real → حقيقي

Imag → تخيلي

$$z = x + jy \Rightarrow \text{Rectangular form} \quad x\text{-}y$$

$$z = R \angle \theta$$
$$= R e^{j\theta}$$

$\Rightarrow$ Polar form θ

كيف نحول من rect إلى polar

$$R = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

كيف تحول من Polar لـ rect

x ، y

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$z = -5 - 3i$$

Rect

conj $\leftarrow$

$$z^{*} = -5 + 3i$$

Polar

$$z = 10\angle 30^\circ$$

$$z^{*} = 10\angle -30^\circ$$

Properties of Complex Numbers

- Addition

+

الآلة حاسبة

حقيقي + حقيقي

تخيلي + تخيلي

$$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$$
$$\rightarrow z = (x_1 + x_2) + j(y_1 + y_2)$$

$$\begin{cases} z_1 = R_1 e^{j\theta_1} \\ z_2 = R_2 e^{j\theta_2} \end{cases}$$
$$\rightarrow z = R_1 e^{j\theta_1} + R_2 e^{j\theta_2}$$

Properties of Complex Numbers

- Subtraction

$$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$$
$$\rightarrow z = (x_1 - x_2) + j(y_1 - y_2)$$

$$\begin{cases} z_1 = R_1 e^{j\theta_1} \\ z_2 = R_2 e^{j\theta_2} \end{cases}$$
$$\rightarrow z = R_1 e^{j\theta_1} - R_2 e^{j\theta_2}$$

Properties of Complex Numbers

آلة حاسبة

- Multiplication

$$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$$
$$\rightarrow z = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$$
$$j^2 = -1$$

$$\begin{cases} z_1 = R_1 e^{j\theta_1} \\ z_2 = R_2 e^{j\theta_2} \end{cases}$$
$$\rightarrow z = (R_1R_2)e^{j(\theta_1+\theta_2)}$$
$$\rightarrow z = (R_1R_2)\angle(\theta_1 + \theta_2)$$

$j = \sqrt{-1}$

$j^2 = -1$

$$Z = (3 + j5)(-5 - 2j)$$

$$j^2 = -1$$

$$Z = -15 - 6j + -25j - j^2 10$$

$$Z = \underline{\underline{-15}} - \underline{\underline{j6}} - \underline{\underline{25j}} + \underline{\underline{10}}$$

$$\boxed{Z = -5 - j31}$$

Properties of Complex Numbers

الآلة حاسبة

- Division

$$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$$

$$\begin{cases} z_1^* = x_1 - jy_1 \\ z_2^* = x_2 - jy_2 \end{cases} \quad \textit{Complex Conjugate}$$

$$\rightarrow z = \frac{z_1}{z_2}$$

$$\rightarrow z = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$\begin{cases} z_1 = R_1e^{j\theta_1} \\ z_2 = R_2e^{j\theta_2} \end{cases}$$

$$\begin{cases} z_1^* = R_1e^{-j\theta_1} \\ z_2^* = R_2e^{-j\theta_2} \end{cases}$$

$$\rightarrow z = \left(\frac{R_1}{R_2}\right)e^{j(\theta_1 - \theta_2)}$$

$$\rightarrow z = \left(\frac{R_1}{R_2}\right)\angle(\theta_1 - \theta_2)$$

Examples

Find j^3 and j^4.

$$j = \sqrt{-1} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = e^{j\frac{\pi}{2}}$$

$$j^3 = \sqrt{-1}\sqrt{-1}\sqrt{-1} = -\sqrt{-1} = -j$$

$$j^3 = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}}$$

$$j^4 = j^3 j = -j^2 = 1$$

Find z^n.

$$z^n = \left(Re^{j\theta}\right)^n = R^n e^{jn\theta} = R^n \angle n\theta$$

$$j^3 = j \cdot j \cdot j = -j$$

$$-1 = j^2, \; j$$

$$j^4 = \boxed{j \cdot j \cdot j \cdot j}$$

$$j^2 \cdot j^2$$

$$= -1 \cdot -1 = \boxed{+1}$$

$$z = 10 \angle 30$$

$$z^5 = (10 \angle 30)^5 = 10^5 \angle 30 \times 5$$

$$\boxed{10^5 \angle 150}$$

لابلاس

Complex Variables

Img

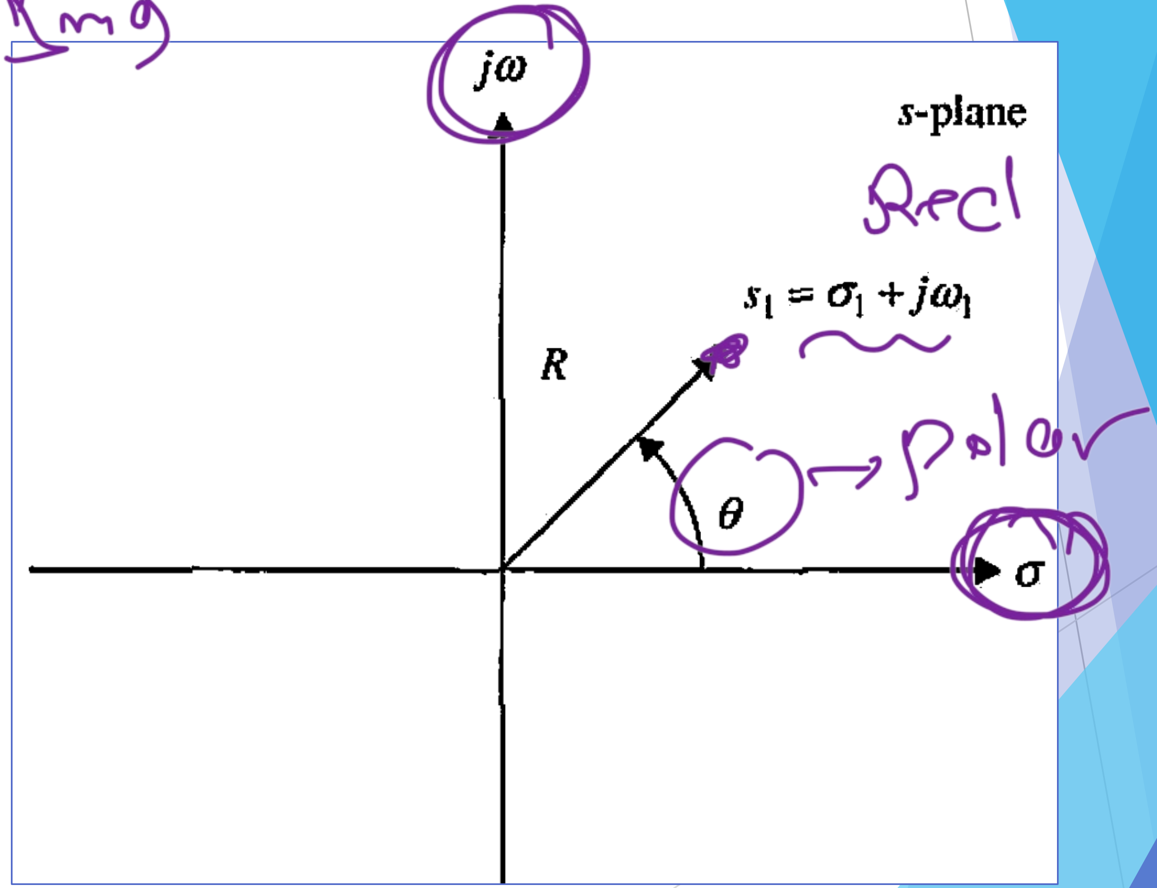
$$s = \sigma + j\omega$$

Real

- Complex s-plane
- Real Axis

  (Horizontal Axis)
- Imaginary Axis

  (Vertical Axis)

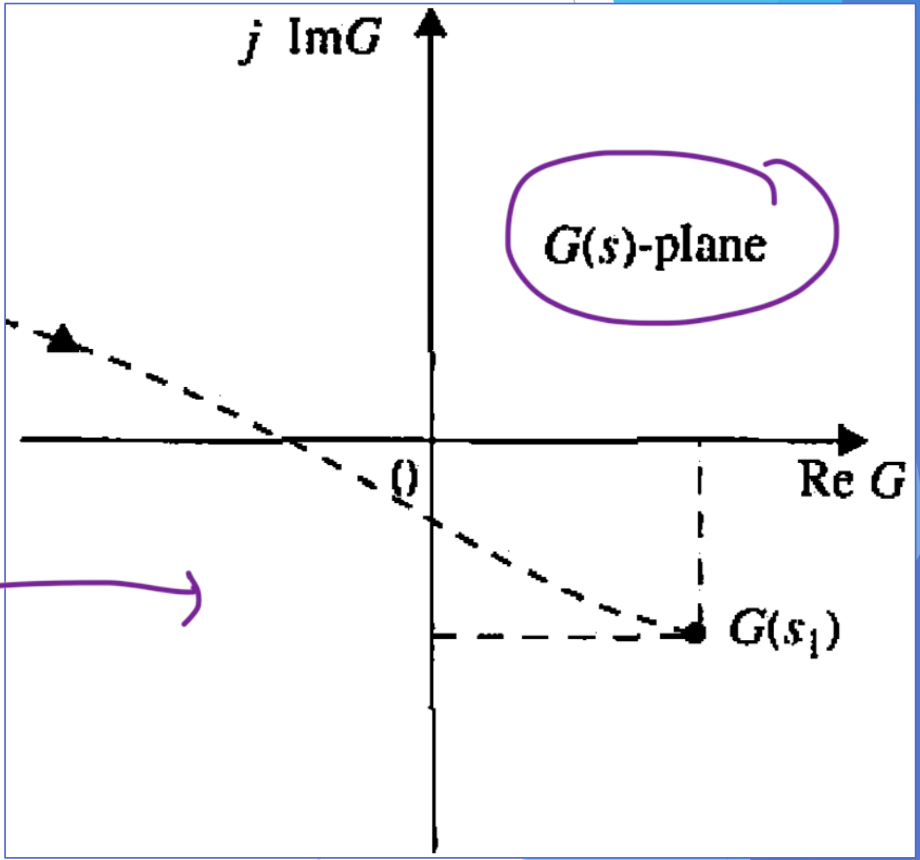


Functions of a Complex Variable

$G(s)$ is a function of complex variable s if for every value of s there is one or more corresponding values of $G(s)$

$$\boxed{G(s) = \mathrm{Re}[G(s)] + j\,\mathrm{Im}[G(s)]}$$

$$\boxed{G(s) = \frac{1}{s(s+1)}}$$



SAAD ARIF (bit.ly/saadarif)

Single-Valued Function

$s=1$
$s=5$

- If for every value of s there is only one corresponding value of $G(s)$ in the $G(s)$-plane then $G(s)$ is a **single-valued function**
- and the mapping from points in the s-plane onto points in the $G(s)$-plane is **single-valued mapping**
- If mapping from the $G(s)$-plane to the s-plane is also single-valued then the mapping is called **one-to-one**

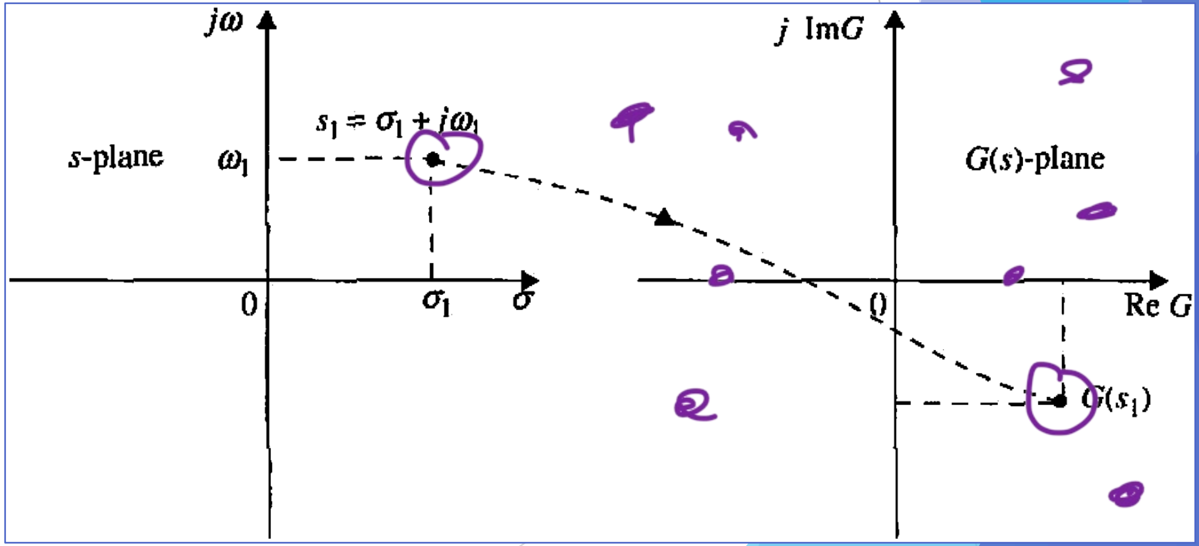
- For many functions, mapping

from function plane to complex

variable plane is not single-valued

e.g.

$$\boxed{G(s) = \frac{1}{s(s+1)}}$$



Analytic Function

A function $G(s)$ of the complex variable s is called **analytic function** in a region of the s-plane if the function and all its derivatives exist in the region

$$G(s) = \frac{1}{s(s+1)}$$

is analytic at every point in the s-plane except $s = 0$ and $s = -1$

$$\frac{1}{0} \quad \text{zero}$$

$$\boxed{G(s) = s + 2}$$

is analytic at every point in the finite s-plane

cont every where

$\ln(x)$

Poles

مشاكل الدالة ⇐ أصفار، مقام ⇐

- The **singularities** of a function are the points in the s-plane at which the function and its derivatives do not exist.
- Most common type of singularity is pole
- If a function $G(s)$ is analytic and single-valued in the neighborhood of point p (has a finite non-zero value) and at point p function is infinite or undefined then $G(s)$ is said to have a **pole** at point p
- In other words, roots of denominator polynomial of $G(s)$ are the poles
- The power of complex variable s in the polynomial defines the order of the pole

  e.g. following function has **simple poles** at $s = 0$ and $s = -1$ and

  a **pole of order 2** at $s = -3$

$$\frac{\text{البسط}}{\text{مقام}}$$

$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^2} \Rightarrow$$

$$\frac{5}{0} \times \qquad \frac{0}{5} \checkmark$$

Zeros

اصفار البسط

لايوجد مثال

- If a function $G(s)$ is analytic and has zero value at a point z then $G(s)$ is said to have a **zero** at point z
- Or if $1/G(s)$ has a pole of order r at $s = z$ then $G(s)$ will have a rth order zero at $s = z$
- In other words, roots of numerator polynomial of $G(s)$ are the zeros
- The power of complex variable s in the polynomial defines the order of the zero

  e.g. following function has **simple zero** at $s = -2$
- For rational function of s, there will be equal number of poles and zeros including poles and zeros at infinity
- Following function has four poles and four zeros including one finite zero and three zeros at infinity

$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^2}$$

عدد التكرار

find Zeros and Poles

O → **Zeros** : أصفار البسط → $s = -2$ } Real

order Zeros $= 1$

X → **Poles** : أصفار المقام $\Rightarrow$

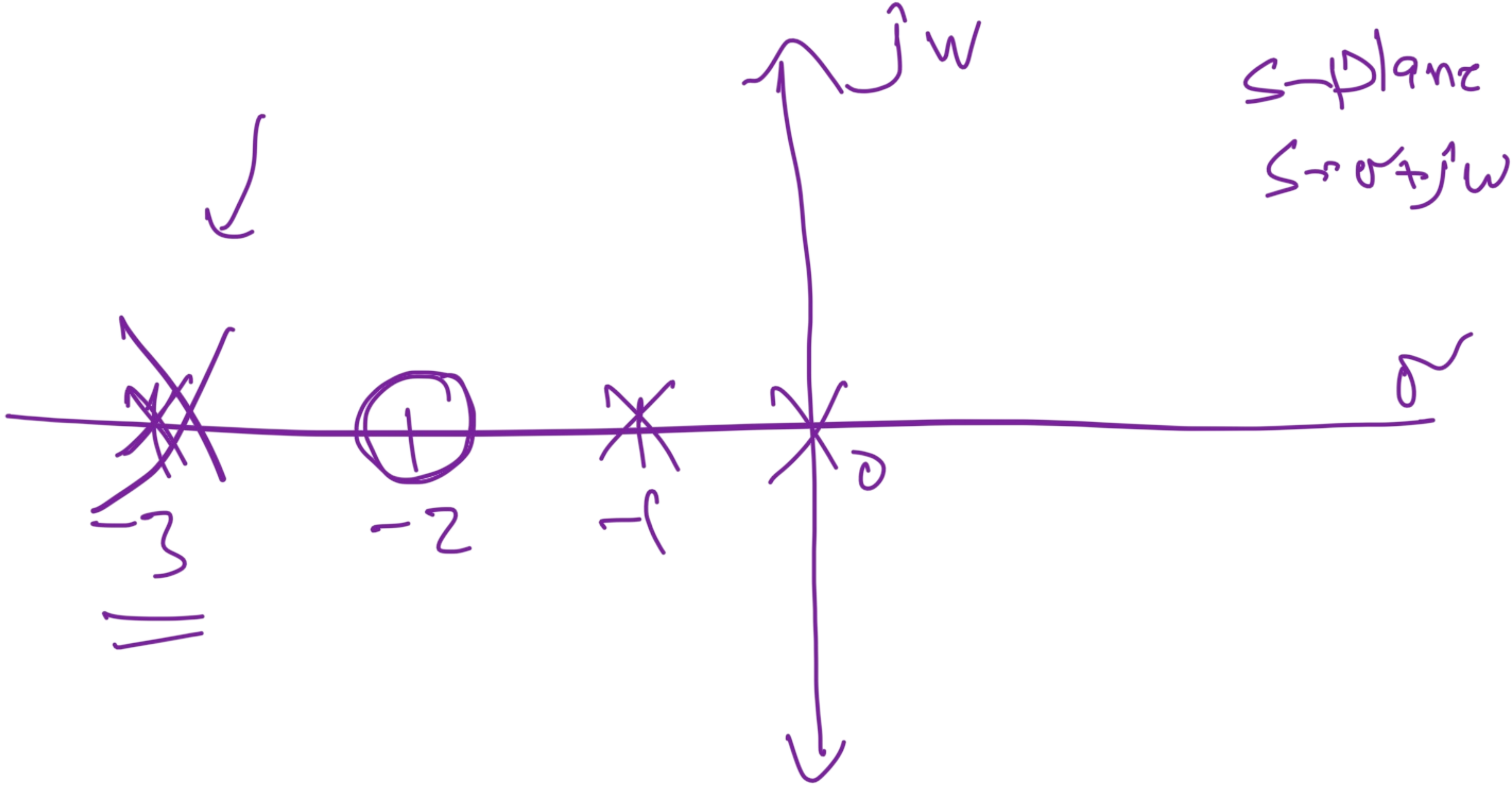
$s = 0$

$s = -1$

$s = -3$

} Real

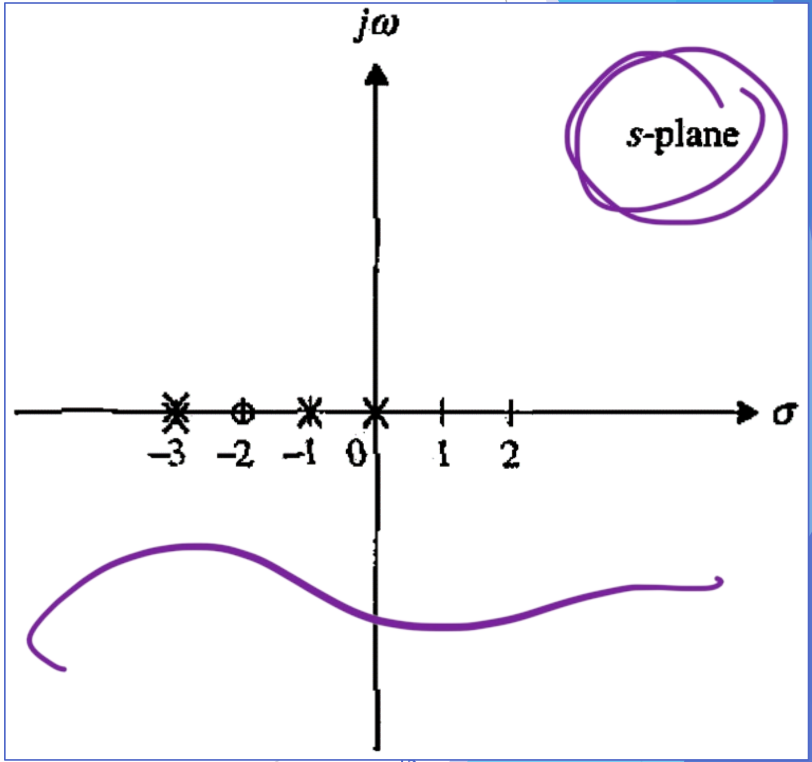
order = 2 ← $s = -3$



s-plane
s = σ + jω
jω
σ
-3
-2
-1
0

Poles and Zeros

- The **poles and zeros** are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics
- Together with the gain constant K they completely characterize the differential equation, and provide a complete description of the system



Differential Equation

RLC series circuit

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = e(t)$$

- Linear Ordinary Differential Equation
  - First Order Linear Ordinary Differential Equation

$$\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

  - Second Order Linear Ordinary Differential Equation

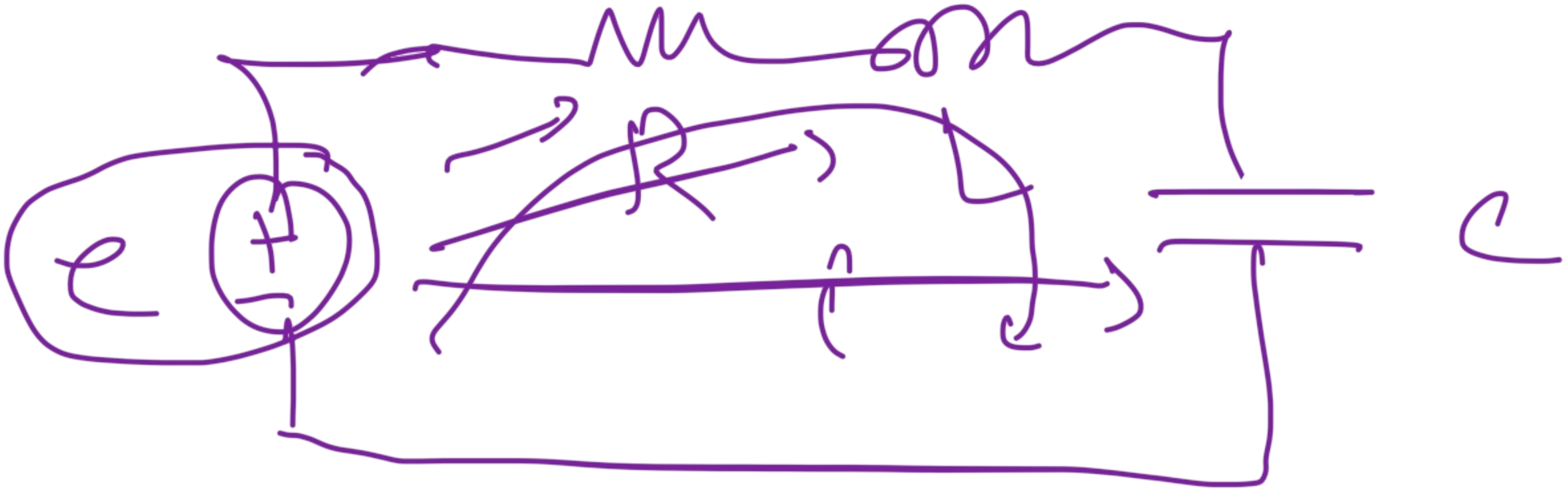
$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

- Non-Linear Ordinary Differential Equation

ارقام

$$m\ell\frac{d^2\theta(t)}{dt^2} + mg\sin\theta(t) = 0$$

Non linear



$$e(t) = V_R + V_L + V_C$$

$$V_R = iR \qquad V_L = L\frac{di}{dt}$$

$$i_C = C\frac{dv}{dt} \Rightarrow V_C = \frac{1}{C}\int i\,dt$$

$$\boxed{e(t) = i(t)R + L\frac{d\,i(t)}{dt} + \frac{1}{C}\int i(t)\,dt}$$

linear

ارقام

x

$$\bigcirc\,\frac{d^2y}{dx^2} + S\frac{dy}{dx} + K y = 0$$

$$y\,y'' + 5y = 0 \Rightarrow \text{non-linear}$$

مشتقة — اس

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^5 + 5y = 0$$

order : 3

Non linear

$$\frac{dy}{dx}$$

Laplace Transform

- Laplace transform is mathematical tool to solve linear ordinary differential equations
- Laplace transform converts the differential equation into an algebraic equation in s-domain
- It is then possible to manipulate the algebraic equation by simple algebraic rules to obtain the solution in the s-domain
- The final solution is obtained by taking the inverse Laplace transform

Theorems of Laplace Transform

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Theorem 1. ***Multiplication by a Constant***
Let k be a constant and $F(s)$ be the Laplace transform of $f(t)$. Then

$$\mathcal{L}[kf(t)] = kF(s)$$

Theorem 2. ***Sum and Difference***
Let $F_1(s)$ and $F_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$, respectively. Then

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

Theorems of Laplace Transform

Theorem 3. ***Differentiation***

Let $F(s)$ be the Laplace transform of $f(t)$, and $f(0)$ is the limit of $f(t)$ as t approaches 0. The Laplace transform of the time derivative of $f(t)$ is

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t \to 0} f(t) = sF(s) - f(0)$$

In general, for higher-order derivatives of $f(t)$,

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - \lim_{t \to 0}\left[s^{n-1} f(t) + s^{n-2}\frac{df(t)}{dt} + \cdots + \frac{d^{n-1} f(t)}{dt^{n-1}}\right]$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

where $f^{(i)}(0)$ denotes the ith-order derivative of $f(t)$ with respect to t, evaluated at $t = 0$.

$$L\left\{\frac{d^{1} f}{dt}\right\} \rightarrow s F(s) - f(0)$$

$$L\left\{\frac{d^{2} f}{dt^{2}}\right\} \rightarrow s^{2} F(s) - s f(0) - f'(0)$$

$$L\left\{\frac{d^{3} f}{dt^{3}}\right\} \rightarrow s^{3} F(s) - s^{2} f(0) - s f'(0) - f''(0)$$

Theorems of Laplace Transform

Theorem 4. *Integration*

The Laplace transform of the first integral of $f(t)$ with respect to t is the Laplace transform of $f(t)$ divided by s; that is,

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

For nth-order integration,

عدد التكاملات

$$\mathcal{L}\left[\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_1} f(t)d\tau dt_1 dt_2\cdots dt_{n-1}\right] = \frac{F(s)}{s^n}$$

Theorems of Laplace Transform

Theorem 5. ***Shift in Time***
The Laplace transform of $f(t)$ delayed by time T is equal to the Laplace transform $f(t)$ multiplied by e^{-Ts}; that is,

$$\mathcal{L}[f(t-T)u_s(t-T)] = e^{-Ts}F(s)$$

where $u_s(t-T)$ denotes the unit-step function that is shifted in time to the right by T.

Theorem 6. ***Initial-Value Theorem***
If the Laplace transform of $f(t)$ is $F(s)$, then

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

if the limit exists.

time domain

t = 0

laplace domain

inverse

Theorems of Laplace Transform

Theorem 7. *Final-Value Theorem*

If the Laplace transform of $f(t)$ is $F(s)$, and if $sF(s)$ is analytic (see Section 2-1-4 on the definition of an analytic function) on the imaginary axis and in the right half of the s-plane, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

The final-value theorem is very useful for the analysis and design of control systems, because it gives the final value of a time function by knowing the behavior of its Laplace transform at $s = 0$. The final-value theorem is *not* valid if $sF(s)$ contains any pole whose real part is zero or positive, which is equivalent to the analytic requirement of $sF(s)$ in the right-half s-plane, as stated in the theorem. The following examples illustrate the care that must be taken in applying the theorem.

Theorems of Laplace Transform

Consider the function

$$F(s) = \frac{5}{s(s^2 + s + 2)}$$

Because $sF(s)$ is analytic on the imaginary axis and in the right-half s-plane, the final-value theorem may be applied. Using Eq. (2-133), we have

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{5}{s^2 + s + 2} = \frac{5}{2}$$

+ 2)

$$F(s) = \frac{5}{s(s^2+s+2)}$$

laplace

$f(0)$

$f(\infty)$

time domin

$$\boxed{\frac{5}{\infty} = 0}$$

$$f(0) = \lim_{s\to\infty} s\,F(s)$$

$$= \lim_{s\to\infty} \cancel{s} * \frac{5}{\cancel{s}(s^2+s+2)} = \boxed{\frac{5}{\infty}}$$

0

$$f(\infty) \Longrightarrow \lim_{s \to 0} s\,F(s)$$

Initial value Thm

$$= \lim_{s \to 0} \cancel{s} \times \frac{5}{\cancel{s}(s^2+s+2)} = \frac{5}{0+0+2}$$

$$\boxed{f(\infty) = \frac{5}{2}} \longrightarrow$$ final value Theorem

Laplace Transform Table

Laplace Transform $F(s)$	Time Function $f(t)$
1	~~Unit~~-impulse function $\delta(t)$
$\frac{1}{s}$	Unit-step function $u_s(t)$
$\frac{1}{s^2}$	Unit-ramp function t
$\frac{n!}{s^{n+1}}$	t^n (n = positive integer)

5 $\quad$ 5 $\delta(t)$

$t^5 \longrightarrow \frac{5!}{s^6}$

t

5! = 5x4x3x2x1

Laplace Transform Table

inverse

$\dfrac{1}{(s+\alpha)(s+\beta)}$	$\dfrac{1}{\beta-\alpha}(e^{-\alpha t}-e^{-\beta t})(\alpha \neq \beta)$
$\dfrac{s}{(s+\alpha)(s+\beta)}$	$\dfrac{1}{\beta-\alpha}(\beta e^{-\beta t}-\alpha e^{-\alpha t})(\alpha \neq \beta)$
$\dfrac{1}{s(s+\alpha)}$	$\boxed{\dfrac{1}{\alpha}(1-e^{-\alpha t})}$
$\dfrac{1}{s(s+\alpha)^2}$	$\dfrac{1}{\alpha^2}(1-e^{-\alpha t}-\alpha t e^{-\alpha t})$

Laplace Transform Table

$\dfrac{1}{s^2(s+\alpha)}$	$\dfrac{1}{\alpha^2}(\alpha t - 1 + e^{-\alpha t})$
$\dfrac{1}{s^2(s+\alpha)^2}$	$\dfrac{1}{\alpha^2}\left[t - \dfrac{2}{\alpha} + \left(t + \dfrac{2}{\alpha}\right)e^{-\alpha t}\right]$
$\dfrac{s}{(s+\alpha)^2}$	$(1-\alpha t)e^{-\alpha t}$

SAAD ARIF (bit.ly/saadarif)

Laplace Transform Table

Laplace Transform $F(s)$	Time Function $f(t)$
$\dfrac{\omega_n}{s^2+\omega_n^2}$	$\sin \omega_n t$
$\dfrac{s}{s^2+\omega_n^2}$	$\cos \omega_n t$
$\dfrac{\omega_n^2}{s(s^2+\omega_n^2)}$	$1-\cos \omega_n t$
$\dfrac{\omega_n^2(s+\alpha)}{s^2+\omega_n^2}$	$\omega_n\sqrt{\alpha^2+\omega_n^2}\,\sin(\omega_n t+\theta)$ where $\theta=\tan^{-1}(\omega_n/\alpha)$

Laplace Transform Table

Design system

$\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t \qquad (\zeta < 1)$
$\dfrac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$	$1 - \dfrac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)$ where $\theta = \cos^{-1}\zeta \qquad (\zeta < 1)$
$\dfrac{s\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$	$\dfrac{-\omega_n^2}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t - \theta\right)$ where $\theta = \cos^{-1}\zeta \qquad (\zeta < 1)$

$$\sin 5t \longrightarrow \frac{5}{s^2 + 5^2 = 25} = \frac{5}{s^2 + 25}$$

$$\cos(7t) \longrightarrow \frac{s}{s^2 + 7^2 = 49} = \boxed{\frac{s}{s^2 + 49}}$$