$$f_{rms} = \sqrt{\frac{1}{T}\int_0^T (f)^2 \, dt}$$

(i.e.) periodic

period

$$V(t) = V_m \sin(\omega t) \quad dt$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T \left[V_m \sin(\omega t)\right]^2 dt}$$

$$V_m \qquad \sqrt{\frac{1}{T}\int_0^T V_m^2 \sin^2(\omega t)\,dt}$$

$$V_{rms} = V_m \sqrt{\frac{1}{T}\int_0^T \underline{\sin^2}(\omega t)\,dt}$$

$$\sin^2 \omega t = \frac{1}{2} - \frac{1}{2}\cos 2\omega t$$

$$\cos^2 \omega t = \frac{1}{2} + \frac{1}{2}\cos 2\omega t$$

---

$$\frac{1}{T}\int_0^T \frac{1}{2} - \frac{1}{2}\cos 2\omega t \; dt$$

$$\frac{1}{T}\left[\frac{1}{2}t - \frac{1}{4}\,\frac{\sin(2\omega t)}{2\omega}\right]_0^T$$

$$\left(\frac{1}{2}T - 0\right)$$

$$\omega = \frac{2\pi}{T} \longrightarrow \omega T = 2\pi$$

$$\sin(\underbrace{2\cdot 2\pi}_{4\pi}) = 0$$

$$\sin(0) = 0$$

$$V_{rms} = V_m \sqrt{\frac{1}{\cancel{T}} \cdot \frac{1}{2} \cancel{T}} = V_m \sqrt{\frac{1}{2}}$$

$$\boxed{V_{rms} = \frac{V_m}{\sqrt{2}}}$$

# AC Circuits and AC Analysis

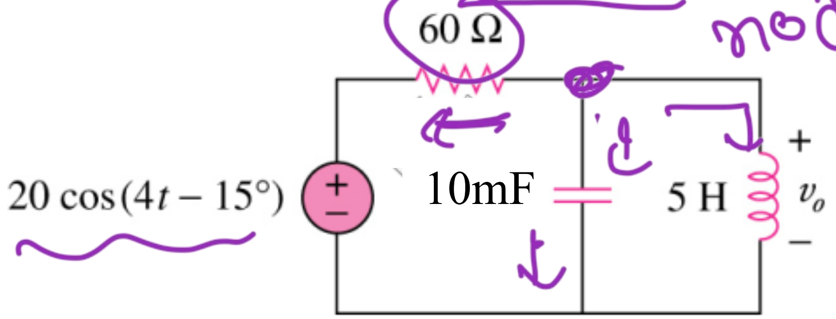
- **Solving AC circuits** in time domain involves solving **differential equations**.
- Example, for the given circuit below, what is Vo(t)?

nodal

Time domain

$i_C = C\frac{dv}{dt}$

$V_L = L\frac{di}{dt}$




- Writing KCL at Vo node results in:

$$\frac{d^2v_o}{dt^2} + \frac{5}{3}\frac{dv_0}{dt} + 20v_0 = -\frac{400}{3}\sin(4t - 15^o)$$

$i_L = \frac{1}{L}\int V_o$

- **However,** forming and solving the DE in time domain are sometimes very tedious.
- **Therefore,** AC circuits analysis is done in **Phasor (frequency) domain**.

$$\frac{V_0 - 20\cos(4t-15)}{60} + 10m\frac{dV_0}{dt} + \frac{1}{5}\int V_0$$

20

$$\frac{V_0}{60} + 10m\frac{dV_0}{dt} + \frac{1}{5}\int V_0\,dt = \frac{1}{3}\cos(4t-15)$$

$$\frac{d^2V_0}{dt^2} + \frac{dV_0}{dt}\boxed{\frac{1}{60\times10\times10^{-3}}} + \frac{1}{5\times10\times10^{-3}}V_0 =$$

$$= \frac{-4}{3\times10^{-3}\times 10}\sin(4t-15)$$

$$\boxed{\frac{d^2v_o}{dt^2} + \frac{5}{3}\frac{dv_o}{dt} + 20v_o = -\frac{400}{3}\sin(4t-15)}$$

# 1-(b) Phasor

# 1-(b) Phasors

- A **phasor** is a **complex number** that represents the amplitude and phase of a sinusoid (**Polar Form**).
- Phasor is the mathematical equivalent of a sinusoid with **time variable dropped**.
- Phasor representation is based on **Euler's identity**.

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \qquad \text{Euler's Identity}$$

$$\cos\phi = \mathrm{Re}\left\{e^{j\phi}\right\} \qquad \text{Real part}$$

$$\sin\phi = \mathrm{Im}\left\{e^{j\phi}\right\} \qquad \text{Imaginary part}$$

- Given a sinusoid $v(t)=V_m\cos(\omega t+\varphi)$.

$$v(t) = V_m\cos(\omega t+\phi) = \mathrm{Re}(V_m e^{j(\omega t+\phi)}) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t}) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

$$\mathbf{V} = V_m e^{j\phi} = V_m\angle\phi = \text{PHASOR REP.}$$

$$v(t) = V_m\cos(\omega t+\phi) \Leftrightarrow \mathbf{V} = \boxed{V_m\angle\phi}$$

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(Time Domain Repr.) (Phasor Domain Representation)

$$v(t) = \mathrm{Re}\{\mathbf{V}e^{j\omega t}\} \qquad \text{(Converting Phasor back to time)}$$

# Complex Numbers and Mathematical Foundation of Phasor

# Complex Numbers

- A complex number may be written in RECTANGULAR FORM as:

RECTANGULAR FORM

$$z = x + jy \quad j=\sqrt{-1},\ \ x=\mathrm{Re}(z),\ \ y=\mathrm{Im}(z)$$

- A second way of representing the complex number is by specifying the MAGNITUDE and $r$ and the ANGLE $\theta$ in POLAR form.

POLAR FORM

$$z = x + jy = |z|\angle\phi = r\angle\phi$$

- The third way of representing the complex number is the EXPONENTIAL form.

EXPONENTIAL FORM

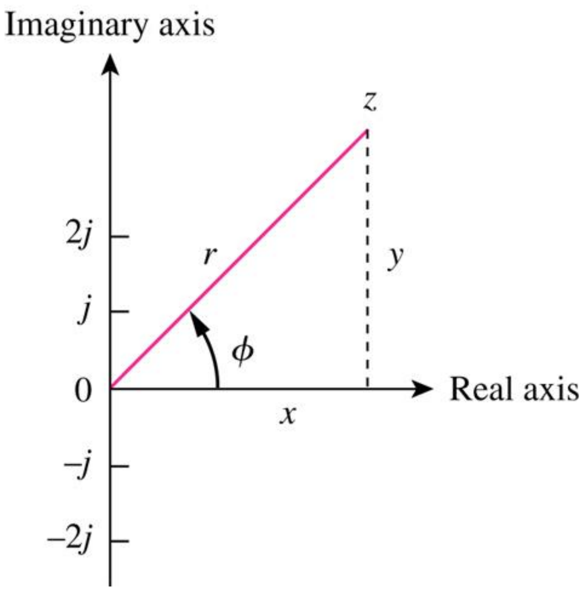
$$z = x + jy = |z|\angle\phi = re^{j\phi}$$

$z = re^{j\phi}$

$$re^{j\phi} = r\cos\phi + jr\sin\phi \quad \text{Euler's Identity}$$

$$r\cos\phi = \mathrm{Re}\left\{re^{j\phi}\right\} \quad \text{Real part}$$

$$r\sin\phi = \mathrm{Im}\left\{re^{j\phi}\right\} \quad \text{Imaginary part}$$



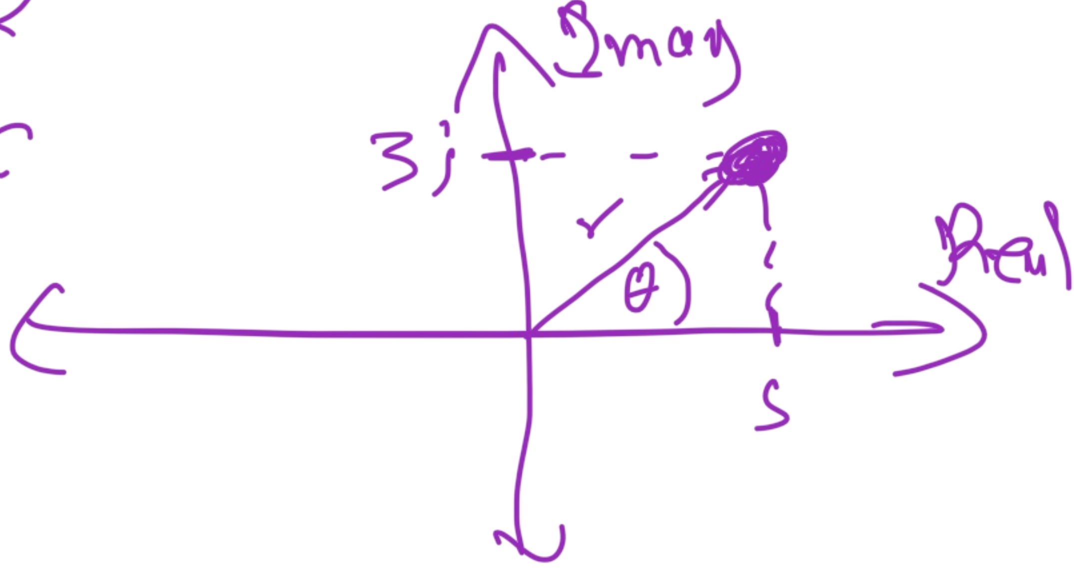

- $x$ is the REAL part.
- $y$ is the IMAGINARY part.
- $r$ is the MAGNITUDE.
- $\varphi$ is the ANGLE.

# Complex number $j = i = \sqrt{-1}$

$$z = \underbrace{x}_{\text{Real part}} + j\underbrace{y}_{\text{imag part}}$$

Rectangular Form

$z = 5 + 3j, \ldots$



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

العدد تخيلي

العدد حقيقي

$$\theta = \tan^{-1}\left(\frac{3}{5}\right)$$

$$Z = r \angle \theta$$

Polar form

direct مش

polar

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$

d! Polar ñão
rect

$$\boxed{z = x + jy}$$

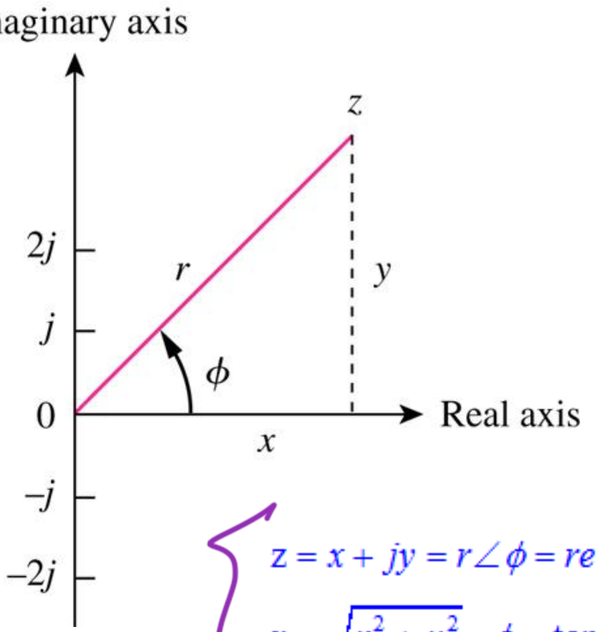
# Complex Number Conversions

- We need to convert COMPLEX numbers from one form to the other form.

$$z = x + jy = r\angle\phi = re^{j\phi} = r(\cos\phi + j\sin\phi)$$

EXPONENTIAL FORM

$$z = x + jy = |z|\angle\phi = re^{j\phi}$$




$$z = x + jy = r\angle\phi = re^{j\phi} = r(\cos\phi + j\sin\phi)$$

$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\dfrac{y}{x}$ Rectangular to Polar

$x = r\cos\phi, \quad y = r\sin\phi$ Polar to Rectangular

# Mathematical Operations of Complex Numbers

- Mathematical operations on complex numbers may require conversions from one form to other form.

ADDITION: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

SUBTRACTION: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Better performed in Rectangular form

MULTIPLICATION: $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$

DIVISION: $\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_2$

RECIPROCAL: $\frac{1}{z} = \frac{1}{r} \angle -\phi$

SQUARE ROOT: $\sqrt{z} = \sqrt{r} \angle \phi/2$

$z = x + jy$

Better performed in Polar form

COMPLEX CONJUGATE: $z^* = x - jy = r\angle -\phi = re^{-j\phi}$

# Complex Numbers

✓ Example 1:

**Evaluate** $[\,(5 + j2)(-1 + j4) - 5\angle 60\,]^*$ and $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ$

Solution:

(a)

$$(5 + j2)(-1 + j4) = -5 + j20 - j2 - 8 = -13 + j18$$
$$5\angle 60^\circ = 2.5 + j4.33$$
$$(5 + j2)(-1 + j4) - 5\angle 60^\circ = -15.5 + j13.67$$
$$[\,(5 + j2)(-1 + j4) - 5\angle 60\,]^* = \underline{\mathbf{-15.5 - j13.67}} = \underline{\mathbf{20.67\angle 221.41^\circ}}$$

(b)

$$3\angle 40^\circ = 2.298 + j1.928$$
$$10 + j5 + 3\angle 40^\circ = 12.298 + j6.928 = 14.115\angle 29.39^\circ$$
$$-3 + j4 = 5\angle 126.87^\circ$$
$$\frac{10 + j5 + 3\angle 40^\circ}{-3 + j4} = \frac{14.115\angle 29.39^\circ}{5\angle 126.87^\circ} = 2.823\angle -97.48^\circ$$
$$2.823\angle -97.48^\circ = -0.3675 - j2.8$$
$$10\angle 30^\circ = 8.66 + j5$$
$$\frac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ = \underline{\mathbf{8.293 + j2.2}}$$

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حاسبة

# Complex Numbers

✓ Example 2:

$$(4\angle 50^\circ)(2\angle -20^\circ) = 4.2\angle\left(50^\circ + (-20^\circ)\right) = 8\angle 30^\circ$$

✓ Example 3:

$$\frac{4\angle 50^\circ}{2\angle -20^\circ} = \frac{4}{2}\angle\left(50^\circ - (-20^\circ)\right) = 2\angle 70^\circ$$

# Back to Phasor

Sinusoids and Phasor

Circuit elements in Phasor domain (frequency domain)

Impedance and admittance

# Phasors

- A **phasor** is a **complex number** that represents the amplitude and phase of a sinusoid (**Polar Form**).
- Phasor is the mathematical equivalent of a sinusoid with **time variable dropped**.
- Given a sinusoid $v(t)=V_m\cos(\omega t+\varphi)$.

Polar

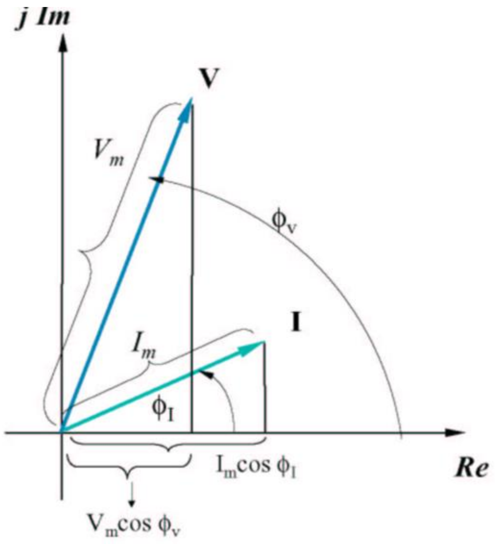
$$v(t) = V_m \cancel{\cos}(\cancel{\omega} t + \phi) \xleftrightarrow{\ \omega\ } \mathbf{V} = \boxed{V_m \angle \phi}$$

(Time Domain Repr.) (Phasor Domain Representation)

# Phasor Diagrams

- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.
- Phasor can be defined using either sine or cosine functions.
- By convention, source of the circuit is used as reference (zero phase angle).
- If not stated, cosine function is used as reference and any voltage or current expression is in the form of a sine is changed to a cosine:

NOTE




Phasor diagram

$+\sin(\omega t) \rightarrow \cos(\omega t - 90)$

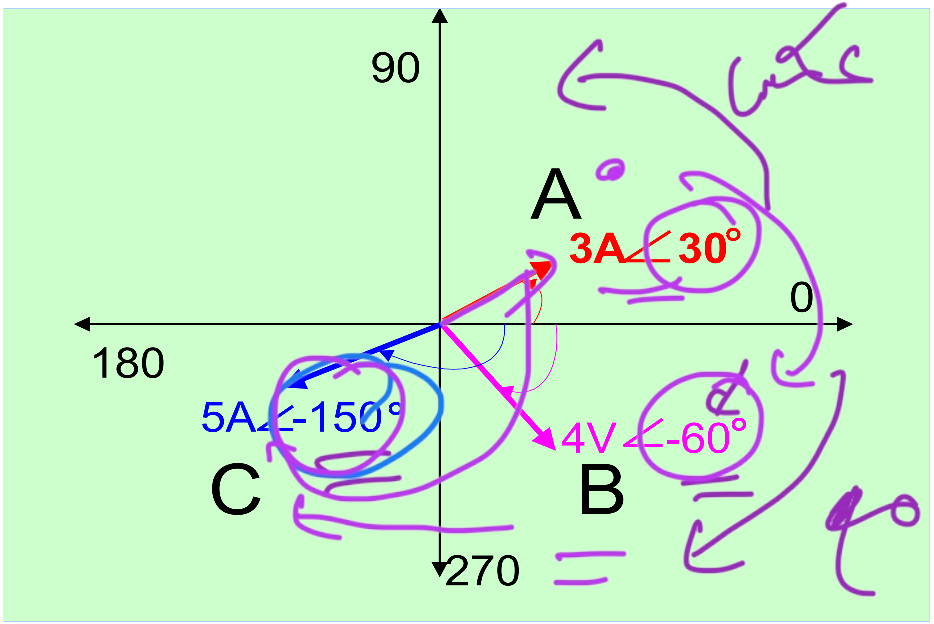
| Time Domain Representation | Phasor Domain Rep. |
|---|---|
| $V_m \cos(\omega t + \phi)$ | $V_m \angle \phi$ |
| $V_m \sin(\omega t + \phi)$ | $V_m \angle \phi - 90°$ |
| $I_m \cos(\omega t + \theta)$ | $I_m \angle \theta$ |
| $I_m \sin(\omega t + \theta)$ | $I_m \angle \theta - 90°$ |

# Phasor Diagrams

عكس عقارب الساعة

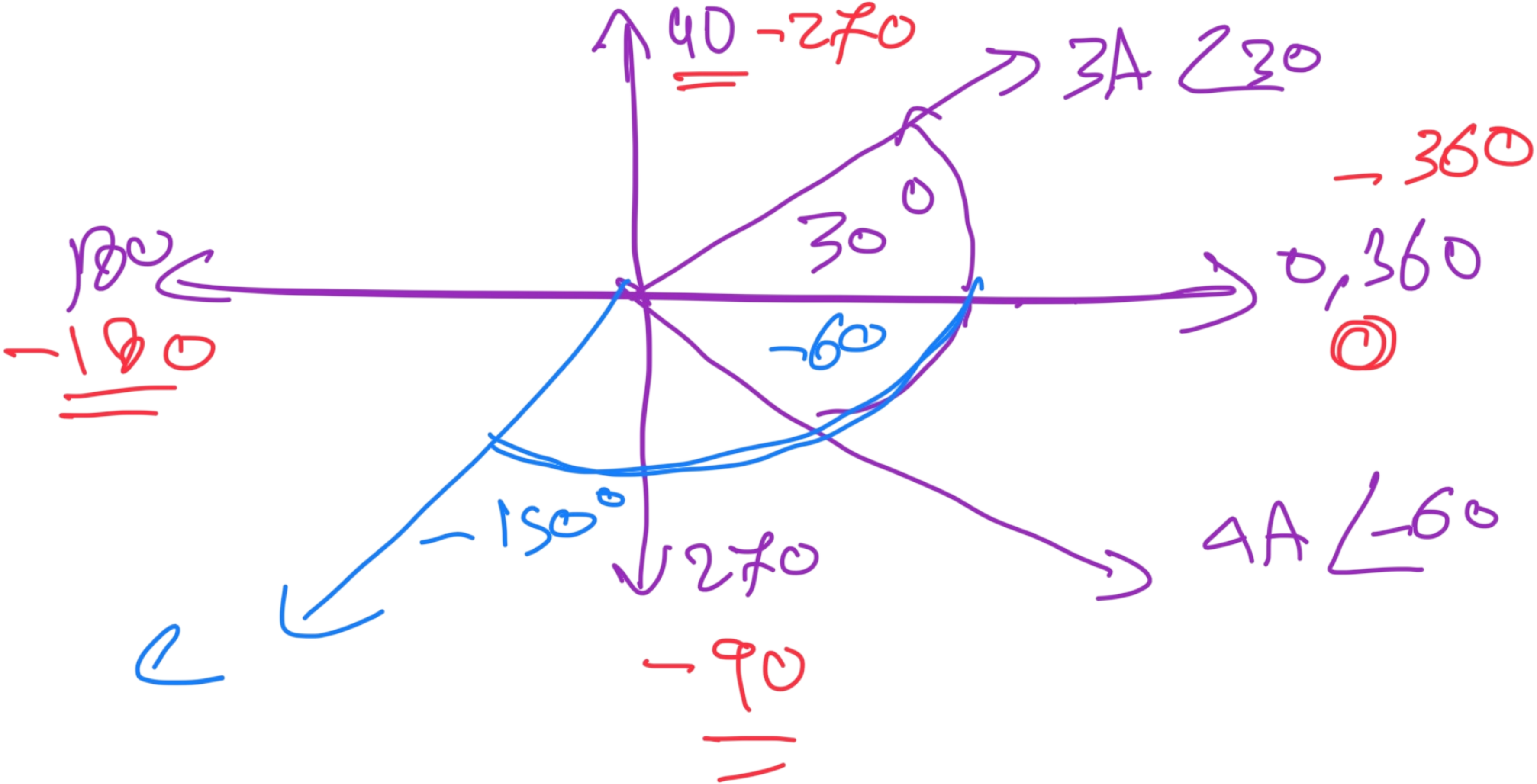
- ✓ Graphing phasors

- ➢ Positive phase angles are drawn counterclockwise from the axis;
- ➢ Negative phase angles are drawn clockwise from the axis. مع عقارب




**Note:**
A leads B
B leads C
C lags A
etc.

A lead C

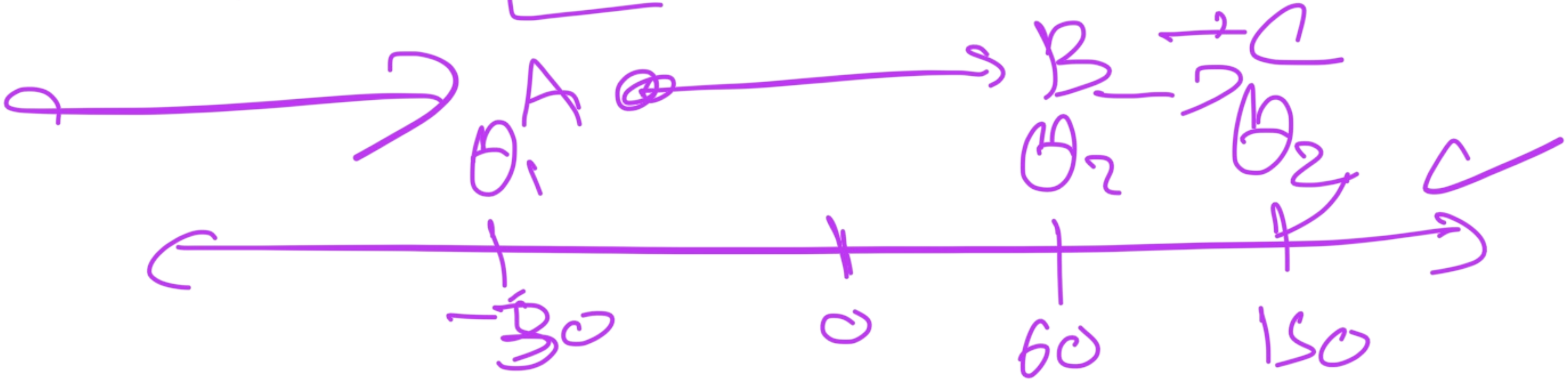


90 -270
3A ∠30°
-360
30°
180°
0,360°
0
-180°
-60°
4A ∠-60°
-150°
270
-90

$A = \angle 30 \rightarrow \theta_1 = -30$

$B = \angle -60 \rightarrow \theta_2 = 60$

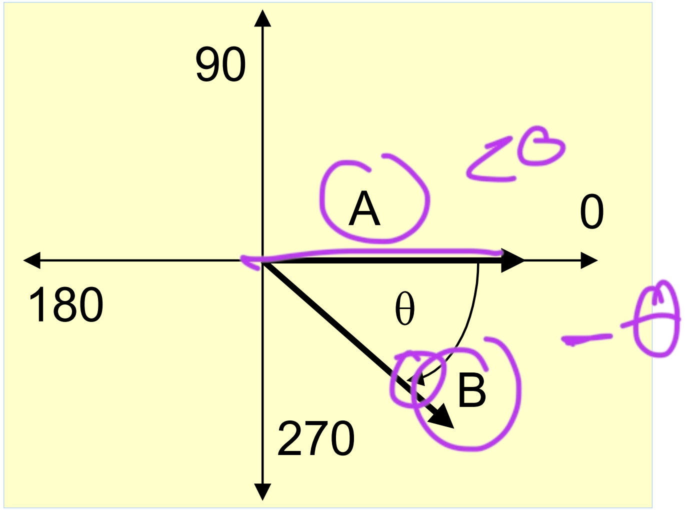
$C = \angle -150 \rightarrow \theta_3 = 150$



# Phasor Diagrams

## ✓ PHASOR DIAGRAM

- Represents one or more sine waves (of the same frequency) and the relationship between them.
- The arrows A and B rotate together. A leads B or B lags A.



# Example 11

Transform the following sinusoids to phasors:

$i = 6\cos(50t - 40^o)$ A $\longrightarrow 6\angle -40$

$v = -4\sin(30t + 50^o)$ V

$+90$

$-\sin(\omega t) \longrightarrow \cos(\omega t + 90)$

$$\sin(\omega t \pm 180^\circ) = -\sin\omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos\omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos\omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin\omega t$$

$V = 4\cos(30t + 140)$

$4\angle 140$

**Solution:**

a. $I = 6\angle -40^\circ$ A

b. Since $-\sin(A) = \cos(A+90^o)$;

$v(t) = 4\cos(30t+50^o+90^o) = 4\cos(30t+140^o)$ V

Transform to phasor => $\boxed{V = 4\angle 140^\circ \text{V}}$

# Time Domain Versus Phasor Domain

**The differences between v(t) and V:**

- v(t) is instantaneous or time-domain representation V is the frequency or phasor-domain representation.
- v(t) is time dependent, V is not.
- v(t) is always real with no complex term, V is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the same frequency.

$\cos(\omega t)$

$\sin(\omega t)$

× لو كانت freq نفسها ⇐ نحول كل الدائرة لـ freq dom ـــ ـــ ـــ

× لو كانت freq مش نفسها


طريقة واحدة فقط ⇐ superposition

# Differentiation and Integration in Phasor Domain

- Differentiating a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$.

$$v(t) = V_m \cos(\omega t + \theta) = \mathrm{Re}\left[\mathbf{V} e^{j\omega t}\right]$$

$$\frac{dv(t)}{dt} = -\omega V_m \sin(\omega t + \theta) = \omega V_m \cos(\omega t + \theta + 90°)$$

which transforms to the phasor

$$\frac{dv(t)}{dt} = \omega V_m e^{j\,(\theta + 90°)} = \omega V_m \underline{/\phi + 90°}$$

$$\frac{dv(t)}{dt} = \omega\, e^{j\,90°}\, V_m e^{j\,\theta}$$

$$\frac{dv}{dt} \Leftrightarrow j\omega \mathbf{V}$$

$$\sin(\omega t \pm 180°) = -\sin \omega t$$
$$\cos(\omega t \pm 180°) = -\cos \omega t$$
$$\sin(\omega t \pm 90°) = \pm \cos \omega t$$
$$\cos(\omega t \pm 90°) = \mp \sin \omega t$$

$$e^{\pm j\phi} = \cos\phi \pm j \sin\phi$$

- Integrating a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$.

# Differentiation and Integration in Phasor Domain


- Relationship between differential, integral operation in phasor listed as follow:

| (Time Domain) | | (Phasor Domain) |
|---|---|---|
| $v(t) = V_m \cos(\omega t + \phi)$ | $\Leftrightarrow$ | $\mathbf{V} = V_m \angle \phi$ |
| $v(t) = V_m \sin(\omega t + \phi)$ | $\Leftrightarrow$ | $\mathbf{V} = V_m \angle \phi - 90^\circ$ |
| $\dfrac{dv}{dt}$ | $\Leftrightarrow$ | $J\omega \mathbf{V}$ |
| $\int v\,dt$ | $\Leftrightarrow$ | $\dfrac{\mathbf{V}}{J\omega}$ |

# Example 9.7: Using Phasor to solve integro-differential equations

**Example 9.7**

Use phasor approach, determine the current *i(t)* in a circuit described by the integro-differential equation.

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

- Solution:

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\underline{/75^\circ}$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\underline{/75^\circ}$$

$$\mathbf{I} = \frac{50\underline{/75^\circ}}{4 - j10} = \frac{50\underline{/75^\circ}}{10.77\underline{/-68.2^\circ}} = 4.642\underline{/143.2^\circ}\ \text{A}$$

Converting this to the time domain,

$$\boxed{i(t) = 4.642\cos(2t + 143.2^\circ)\ \text{A}}$$

| | | |
|---|---|---|
| $v(t)$ | $\longleftrightarrow$ | $V = V\angle\phi$ |
| $\dfrac{dv}{dt}$ | $\longleftrightarrow$ | $j\omega V$ |
| $\int v\,dt$ | $\longleftrightarrow$ | $\dfrac{V}{j\omega}$ |

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75)$$

$w = 2$

Jop $\rightarrow$ freq domain

$$4I + \frac{8I}{jw} - 3Ijw = 50\angle 75$$

$$4I - j4I - 6jI = 50\angle 75$$

$$I(4 - 10j) = 50\angle 75$$

$$I = \frac{50 \angle 75}{4 - 10j} = -3{,}717 + j\,2{,}781$$

$$\boxed{I = 4{,}642 \angle 143{,}2 \text{ A}}$$

$$\boxed{i(t) = 4{,}642 \cos(2t + 143{,}2)} \text{ A}$$

# Phasor Relationships for Circuit Elements

- Ohm's Law still applies even though the voltage source is AC

$\frac{\partial}{\partial t} \to j\omega$

$\int \to \frac{1}{j\omega}$

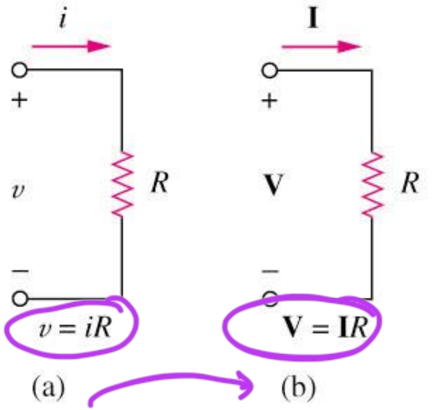
**Terminal Equations**

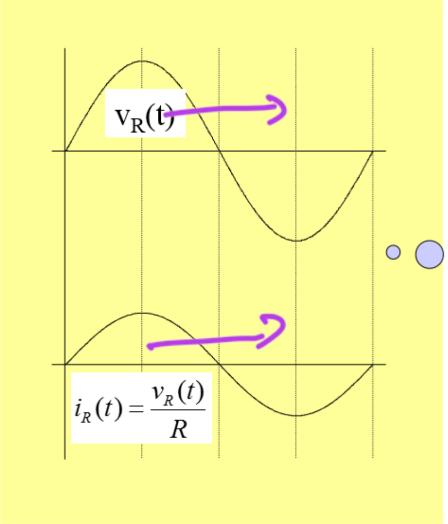
4

| Time Domain | Frequency Domain |
|---|---|
| R, $i(t)$, + $v(t)$ − : $v(t) = Ri(t)$ | R, $\hat{I}$, + $\hat{V}$ − : $\hat{V} = R\hat{I}$, $\hat{I} = G\hat{V}$ |
| L, $i(t)$, + $v(t)$ − : $v(t) = L\dfrac{di(t)}{dt}$ <br> $i_L(t) = \frac{1}{L}\int i\,dt$ | $j\omega L$, $\hat{I}$, + $\hat{V}$ − : $\hat{V} = j\omega L\hat{I}$, $\hat{I} = \dfrac{1}{j\omega L}\hat{V}$ |
| C, $i(t)$, + $v(t)$ − : $i(t) = C\dfrac{dv(t)}{dt}$, $v(t) = \dfrac{1}{C}\int i(t)dt$ | C, $\hat{I}$, + $\hat{V}$ − : $\hat{V} = \dfrac{1}{j\omega C}\hat{I}$, $\hat{I} = j\omega C\hat{V}$ |

# Phasor Relationships for Circuit Elements

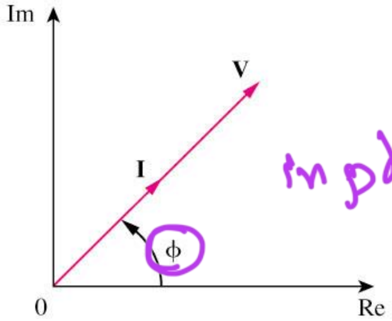
## A- Resistor:



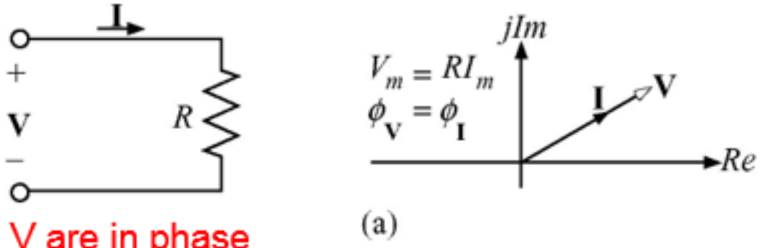




**PHASE ANGLE FOR R, θ=0°**

shift




in phase

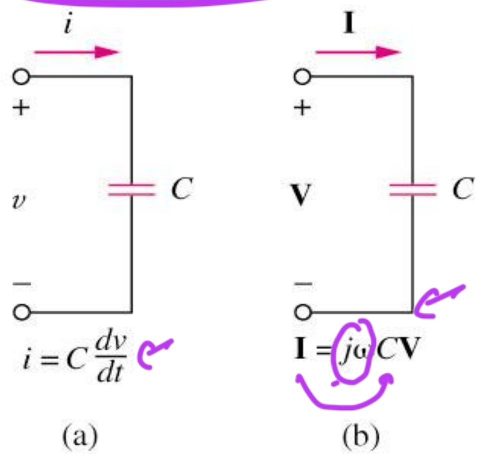



For R, I and V are in phase

# Phasor Relationships for Circuit Elements

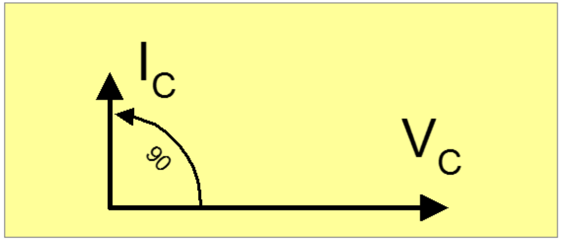
V lag I 90
I lead V

## B- Capacitor:

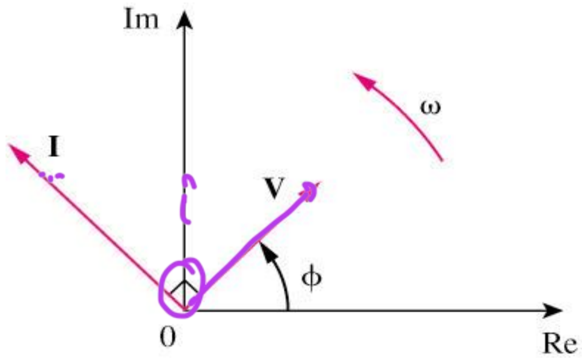


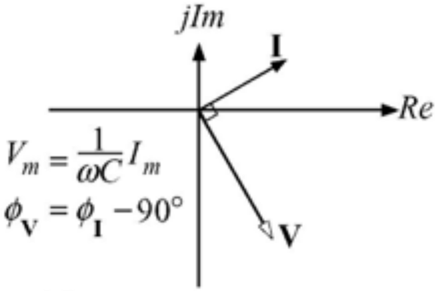

- In the Capacitor (C), Voltage **LAGS** charging current by 90° <u>or</u> Charging Current (I) **LEADS** Voltage (E) by 90°

- <u>**I. C. E.**</u>




$j \rightarrow 1\angle 90$

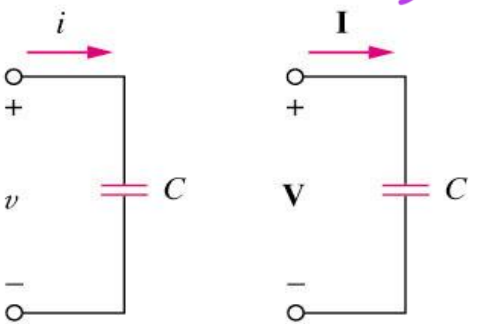


# Phasor Relationships for Circuit Elements

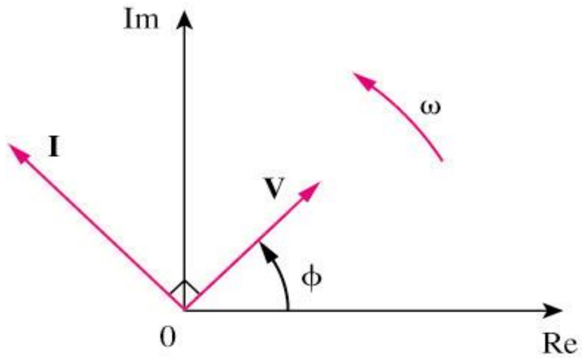
## B- Capacitor:

$\frac{1}{j} = -j$




$i = C\frac{dv}{dt}$ $\qquad$ $\mathbf{I} = j\omega C\mathbf{V}$ $\rightarrow \mathbf{V} = -j \underbrace{\boxed{\frac{1}{\omega C}}}_{}^{X_C} \mathbf{I}$

(a) $\qquad$ (b)




### ❖ CAPACITIVE REACTANCE

- In resistor, the Ohm's Law is V=IR, where R is the opposition to current.
- We will define Capacitive Reactance, $X_C$, as the opposition to current in a capacitor.

$$V = I\,X_C$$

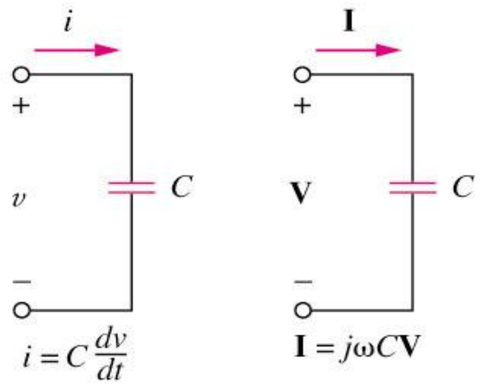
- $X_C$ will have units of Ohms.
- Note inverse proportionality to *f* and C.

$$X_C = \frac{1}{2\pi f C} = \frac{1}{\omega C}$$

Magnitude of $X_C$

# Phasor Relationships for Circuit Elements

## B- Capacitor:

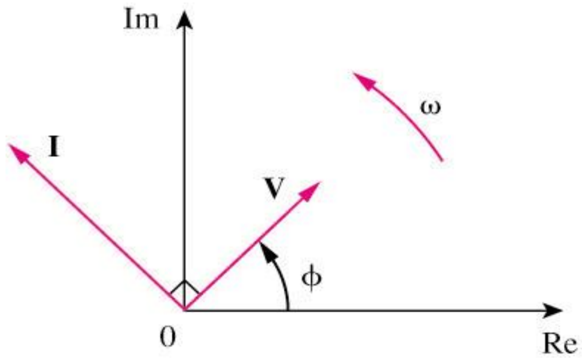



$i = C\frac{dv}{dt}$

(a)

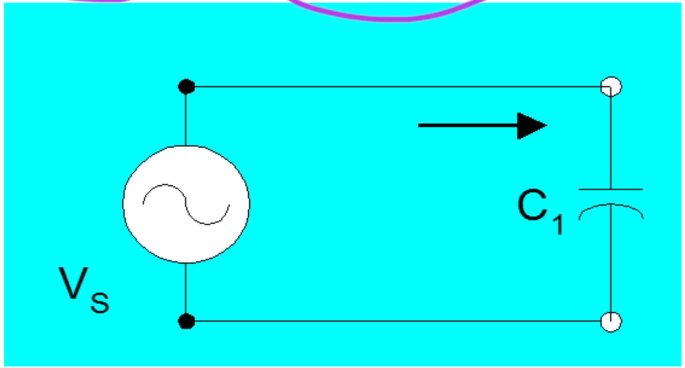
$\mathbf{I} = j\omega C\mathbf{V}$

(b)




❖ **CAPACITIVE REACTANCE**

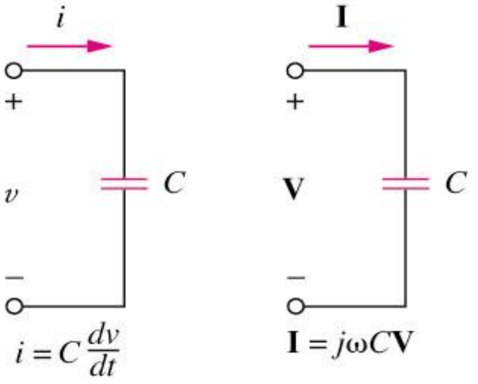
Ex: $f$ = 500 Hz, C = 50 μF, $X_C$ = ?



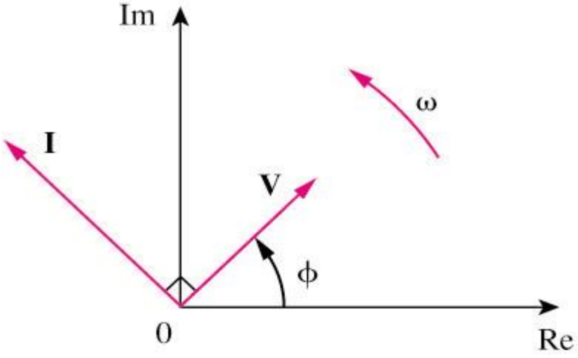

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi(500\,Hz)50\,\mu F} = 6.4\Omega$$

# Phasor Relationships for Circuit Elements

## B- Capacitor:




(a) (b)




## ❖ PHASE ANGLE FOR $X_C$

- Capacitive reactance also has a phase angle associated with it.

$$\mathbf{X}_C = \frac{\mathbf{V}}{\mathbf{I}}$$

$$\mathbf{X}_C = \frac{V\angle 0°}{I\angle 90°} = Z\angle -90°$$

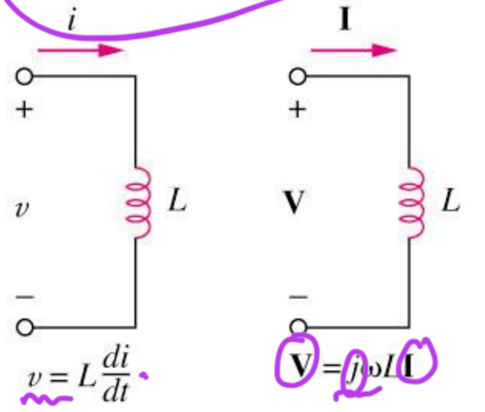
- The phase angle for Capacitive Reactance ($\mathbf{X}_C$) will always = -90°
- $\mathbf{X}_C$ may be expressed in POLAR or RECTANGULAR form.

$$X_C\angle -90° \quad \text{or} \quad -jX_C$$

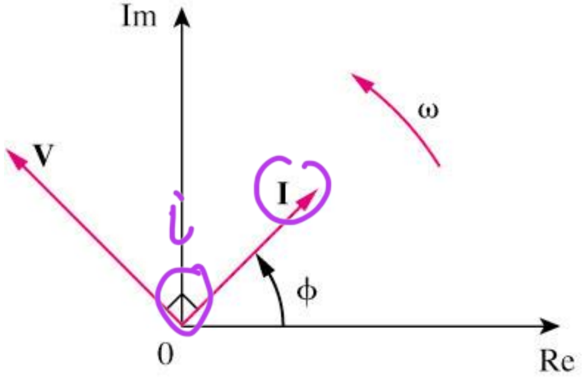
$$\boxed{Z_C = -jX_C} \quad \& \quad Z_C = \frac{1}{j\omega C}$$

# Phasor Relationships for Circuit Elements

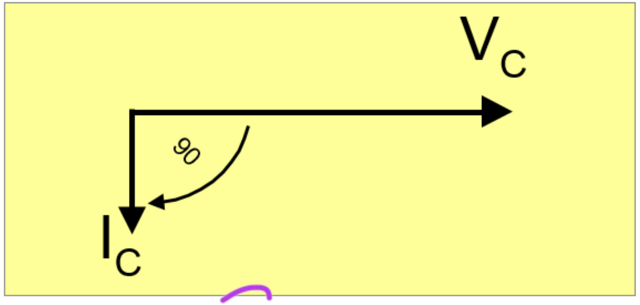
## C-Inductor:




(a) (b)

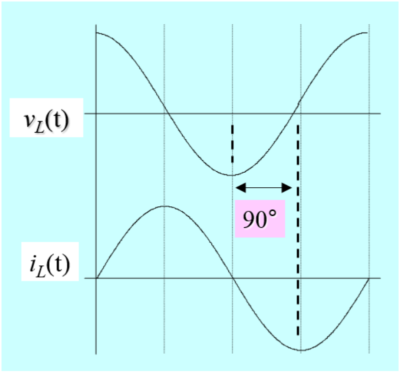


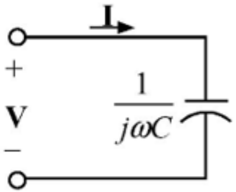

- In the Inductor (L), Induced Voltage LEADS current by 90° or Current (I) LAGS Induced Voltage (E) by 90°.



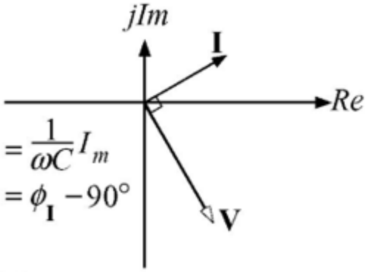

V lead I by 90
I lag V

**Graph $v_L(t)$ and $i_L(t)$**




For C, I Leads V by 90°



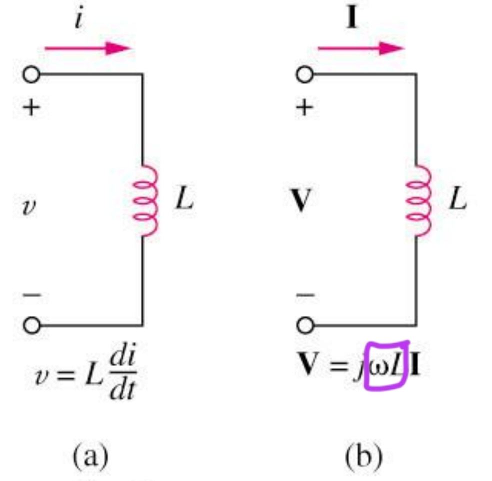

$V_m = \dfrac{1}{\omega C} I_m$

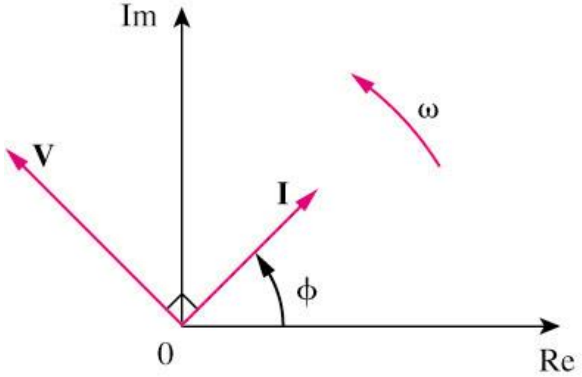
$\phi_{\mathbf{V}} = \phi_{\mathbf{I}} - 90°$

(c)

# Phasor Relationships for Circuit Elements

## C-Inductor:




❖ **INDUCTIVE REACTANCE**

❖ We will define Inductive Reactance, $X_L$, as the opposition to current in an inductor.

$$V = I X_L$$

X.

V = ○ I

❖ $X_L$ will have units of Ohms (Ω).

❖ Note direct proportionality to f and L.

$$X_L = 2\pi f L = \omega L$$

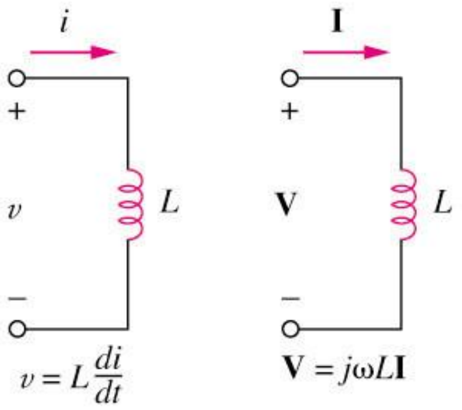
Magnitude of $X_L$

$V = X_C\, I$ $\qquad \frac{1}{\omega C}$

$V = X_L\, I$ $\qquad \omega L$

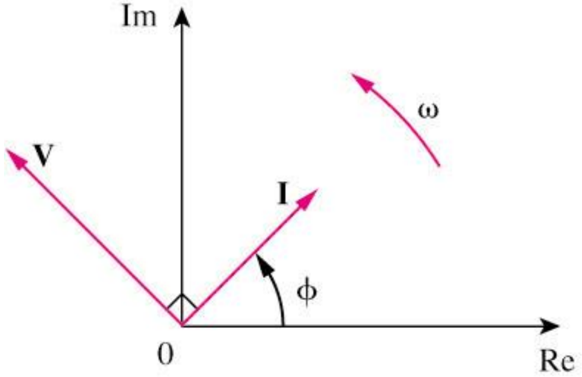
# Phasor Relationships for Circuit Elements

## C-Inductor:



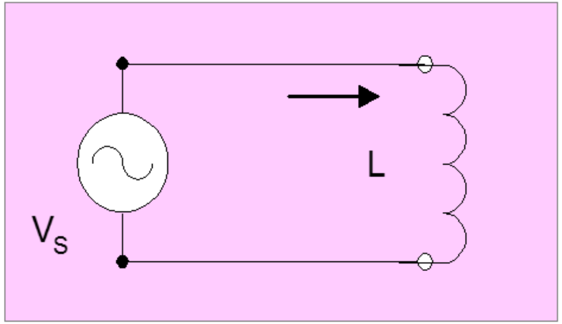

$v = L\frac{di}{dt}$ $\mathbf{V} = j\omega L\mathbf{I}$

(a) (b)




❖ **INDUCTIVE REACTANCE**

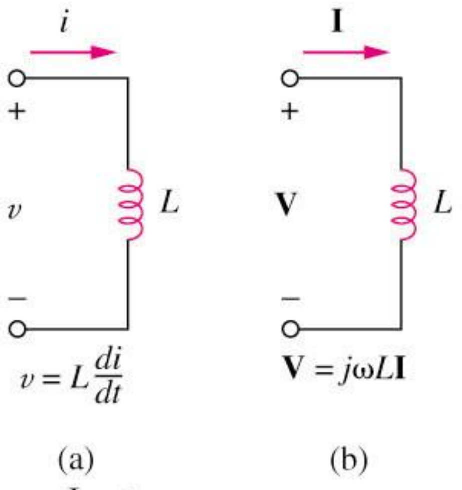
$f$ = 500 Hz, C = 500 mH, $X_L$ = ?

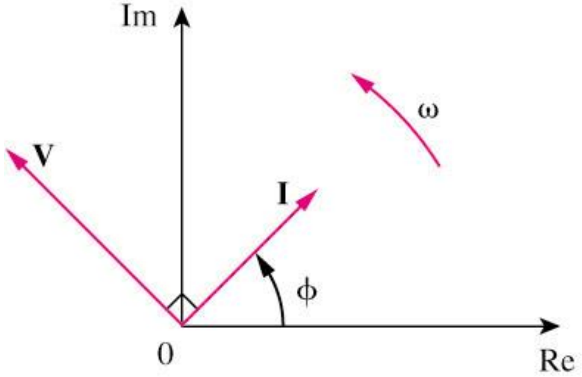



$$X_L = 2\pi fL = 2\pi(500)(.5) = 1.57k\Omega$$

# Phasor Relationships for Circuit Elements

## C-Inductor:




❖ **PHASE ANGLE FOR $X_L$**

❖ If $\underline{V}$ is our reference wave:

$$\mathbf{X}_L = \frac{V\angle 0°}{I\angle -90°} = Z\angle +90°$$

- The phase angle for Inductive Reactance ($\mathbf{X}_L$) will always = +90°
- $\mathbf{X}_L$ may be expressed in POLAR or RECTANGULAR form.

$$X_L\angle 90° \quad \text{or} \quad jX_L \quad \Omega$$

$$\boxed{Z_L = jX_L} \rightarrow \boxed{Z_L = j\omega L}$$

# Phasor Relationships for Circuit Elements


✓ **COMPARISON OF $X_L$ & $X_C$**

- $X_L$ is directly proportional to frequency and inductance.

$$X_L = 2\pi f L = \omega L$$

- $X_C$ is inversely proportional to frequency and capacitance.

$$X_C = \frac{1}{2\pi f C} = \frac{1}{\omega C}$$

# Phasor Relationships for Circuit Elements

$\infty \Rightarrow$ open

$0 \Rightarrow$ short

DC

- ❖ **Frequency effects**

    - Using the reactances of an inductor and a capacitor you can show the effects of low and high frequencies on them.

DC

- At low frequencys (f=0):
    - an inductor acts like a short circuit.
    - a capacitor acts like an open circuit.
- At high frequencys (f=∞):
    - an inductor acts like an open circuit.
    - a capacitor acts like a short circuit.

$$X_C = \frac{1}{2\pi f C} = \frac{1}{\omega C}$$

$$X_L = 2\pi f L = \omega L$$

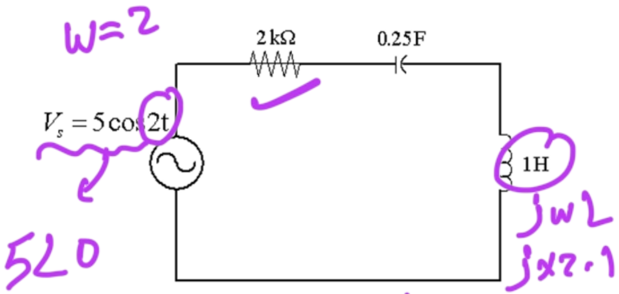
$\frac{1}{0}$ $\infty$

$\boxed{\infty}$

$\frac{1}{\infty} \to \boxed{0}$

# Example

**Represent the below circuit in frequency domain:**

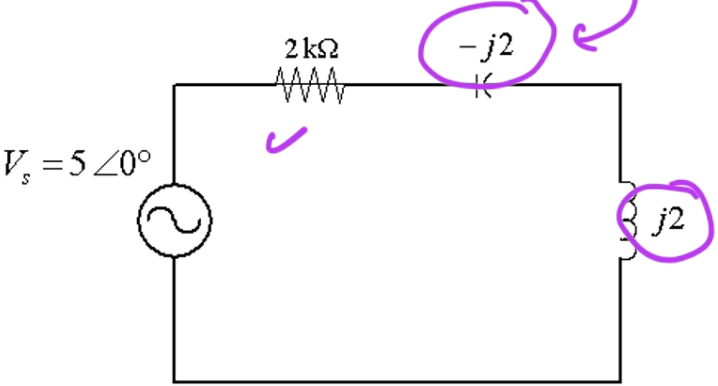



## Solution:

reactance $X_L$ =
$X_C$ =

- ω=2 rad/s:

| Time domain | Freq domain |
|---|---|
| R = 2 Ω | R = 2 Ω |
| C = 0.25 F | $\cancel{X}Z_C = -j(1/\omega C) = -j2\ \Omega$ |
| L = 1 H | $\cancel{X}Z_L = j\omega L = j2\ \Omega$ |
| $V_s$ = 5 cos 2t | $V_s$ = 5<0° |



# Impedance

$$Z_R = R \quad \Omega$$

$$Z_L = j\omega L \quad \Omega$$

$$Z_C = \frac{1}{j\omega C} \quad \Omega$$

# Example 9.8

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

**Solution:**

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\underline{/45°}$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45°}}{j60 \times 0.1} = \frac{12\underline{/45°}}{6\underline{/90°}} = 2\underline{/-45°}\text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°)\text{ A}$$

$$v(t) = 12\cos(60t + 45) \qquad L = 0.1$$

$$i(t) = \frac{1}{2}\int v(t)\,dt$$

$$V = 12\angle 45 \text{ V}$$

Freq دل التمثيل

dom

$$w = 60$$

$$Z_L = jwL = j \times 60 \times 0.1 = 6j\ \Omega$$

$$V = IZ \longrightarrow I = \frac{V}{Z}$$

$$I = \frac{12\angle 45}{6j} = 2\angle -45\ A$$

$$\boxed{i(t) = 2\cos(60t - 45)\ A}$$


# Using phasor to add sinusoids- Example 9.6

Given $i_1(t) = 4\cos(\omega t + 30°)$ A and $i_2(t) = 5\sin(\omega t - 20°)$ A find their sum.

**Solution:**

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\underline{/30°}$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$\boxed{i_2(t) = 5\sin(\omega t - 20°) = 5\cos(\omega t - 20° - 90°) = 5\cos(\omega t - 110°)}$$

and its phasor is

$$\mathbf{I}_2 = 5\underline{/-110°}$$

$$\sin(\omega t \pm 180°) = -\sin\omega t$$
$$\cos(\omega t \pm 180°) = -\cos\omega t$$
$$\sin(\omega t \pm 90°) = \pm\cos\omega t$$
$$\boxed{\cos(\omega t \pm 90°) = \mp\sin\omega t}$$

If we let $i = i_1 + i_2$, then

$$\begin{aligned}\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 &= 4\underline{/30°} + 5\underline{/-110°}\\ &= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698\\ &= 3.218\underline{/-56.97°}\text{ A}\end{aligned}$$

$i_1(t) = \underline{\underline{4\cos}}(wt + 30)$ ✓

$i_1(t) + i_2(t)$

$i_2(t) = \underline{\underline{5\sin}}(wt - 20) - 90$

$\hookrightarrow \underline{\underline{5\cos}}(wt - 110)$

Polar $\leftarrow$ Freq dom

$$\left.\begin{aligned} i_1 &= 4\angle 30 \\ i_2 &= 5\angle -110 \end{aligned}\right\}$$

$$i_T = i_1 + i_2 = (4\angle 30) + (5\angle -110)$$

$$3{,}218\angle -56{,}98$$

$$\boxed{i_T(t) = 3{,}218\cos(\omega t - 56{,}98)\ A}$$

# Impedance and Admittance

# A-Impedance

$R = \frac{V}{I}$

- The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms Ω.

$$Z = \frac{V}{I} = R + jX$$

- Impedance is a complex quantity:
  - R = Real part of Z = Resistance
  - X = Imaginary part of Z = Reactance

$X_L, X_C$

| Impedances of passive elements | |
|---|---|
| **Element** | **Impedance** |
| **R** | $Z = R$ |
| **L** | $Z = j\omega L$ |
| **C** | $Z = \frac{1}{j\omega C} = \frac{-j}{\omega C}$ |

- Impedance in polar form:

$$Z = \frac{V}{I} = R + jX = |Z|\angle\theta$$

$$|Z| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |Z|\cos\theta \qquad X = |Z|\sin\theta$$

- The Reactance may be positive or negative:
  - It is Inductive if $X$ is positive.
  - It is Capacitive if $X$ is negative.

$+jX \rightarrow X_L \rightarrow L$

$-jX \rightarrow X_C \rightarrow C$

# A-Impedance

## IMPEDANCES SUMMARY

| Impedance | Phasor form: | Rectangular form |
|---|---|---|
| $Z_R$ | $R\angle 0^o$ | R+j0 |
| $Z_L$ | $X_L\angle 90^o$ | $0+jX_L$ |
| $Z_C$ | $X_C\angle -90^o$ | $0-jX_C$ |

$Z_L = j\omega L = jX_L$

$Z_C = \frac{-j}{\omega C} = -jX_C$

$X_C = \frac{1}{\omega C}$

$X_L = \omega L$

# B- ADMITTANCE

- ✓ The reciprocal of impedance.
- ✓ Symbol is Y.
- ✓ Measured in Siemens (S).

$$Y = \frac{1}{Z} = \frac{I}{V}$$

➢ Admittance is a complex quantity:

$$\mathrm{Y} = \mathrm{G} \pm j\mathrm{B}$$

- G = Real part of Y : Conductance
- B = Imaginary part of Y : Susceptance


**Z AND Y OF PASSIVE ELEMENTS**

| ELEMENT | IMPEDANCE (Z) | ADMITTANCE (Y) |
|---|---|---|
| R | $Z = R$ | $\mathrm{Y} = \frac{1}{\mathrm{R}}$ |
| L | $Z = j\omega L$ | $\mathrm{Y} = \frac{1}{j\omega L}$ |
| C | $Z = -j\frac{1}{\omega C}$ | $\mathrm{Y} = j\omega C$ |

$\frac{1}{j\omega C}$

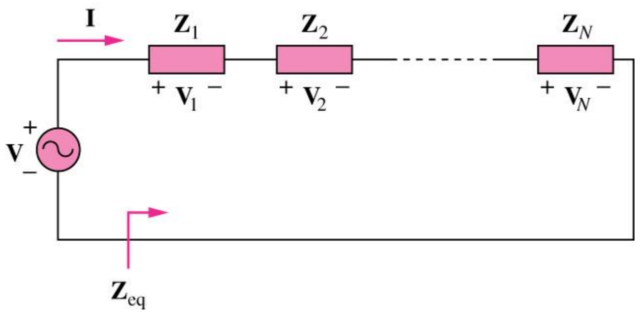
$$Y = \frac{1}{Z} = \frac{1}{R + jX} = G + jB$$

$$\cancel{\neq \frac{1}{R} + \frac{1}{jX}}$$

# C-TOTAL IMPEDANCE FOR AC CIRCUITS ⇒ Resistance

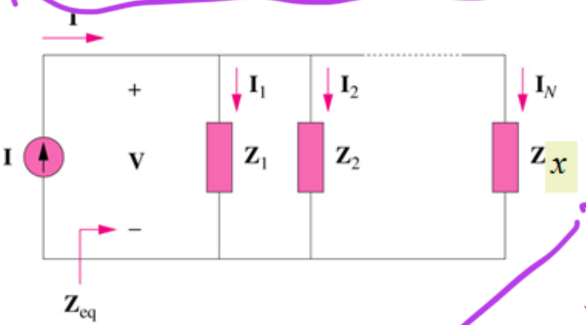
- To compute total circuit impedance in AC circuits, use the same techniques as in DC.
- The only difference is that instead of using resistors, you now have to use **complex impedance, Z.**

### ❖ For series connected impedances:




$$Z_{eq} = \frac{V}{I} = Z_1 + Z_2 + \cdots + Z_N \quad \text{(Equivalent Impedance)}$$

### ❖ Total impedance for parallel circuit:




$$\frac{1}{\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\cdots\right)}$$

$$\frac{1}{Z_{total}} = \sum \frac{1}{Z_x} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_x} \Rightarrow$$

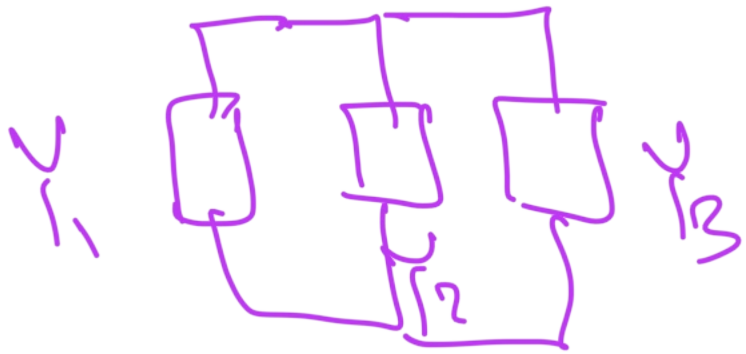
$$Z_{total} = \left(\sum \frac{1}{Z_x}\right)^{-1} = \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_x}\right)^{-1}$$

- The impedance can be easily computed from the admittance:

$$Z_{total} = \frac{1}{Y_{total}} = \left[Y_{total}\right]^{-1}$$

$$\therefore Y_{total} = \left(Y_1 + Y_2 + \ldots + Y_x\right)$$


$$\boxed{Z_T = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}}}$$




$$\rightarrow Y_T = Y_1 + Y_2 + Y_3$$

$$Z_T = \frac{1}{Y_{tot}}$$

$Y_1 = \frac{1}{Z_1}$ $Y_2 = \frac{1}{Z_2}$ $Y_3 = \frac{1}{Z_3}$

# C-TOTAL IMPEDANCE FOR AC CIRCUITS

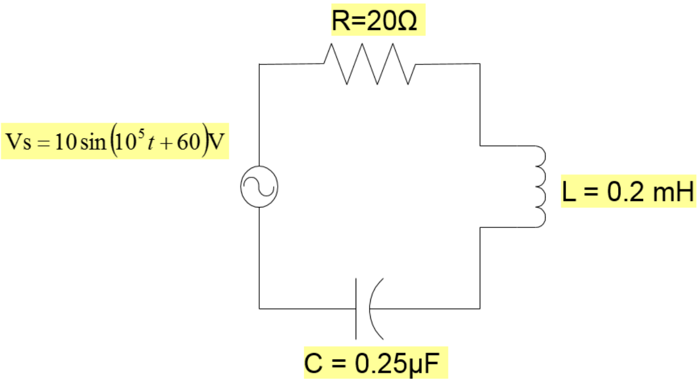
## 1-Example of series circuit:

a- Find Total Impedance

b- Draw Impedance Triangle (Phasor Diagram).

c- Find $i_s$, $v_R$, $v_C$, $v_L$

d- Find $v_R$, $v_C$, $v_L$ using Voltage Divider

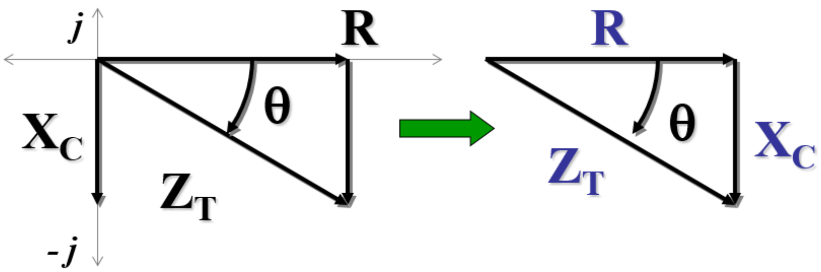



✓ **Solution**

➢ a- The total Impedance

$$
\begin{aligned}
Z_T &= \sum Z_R + Z_L + Z_C \\
&= 20 + j20 + (-j40) \\
&= (20 - j20)\Omega \\
&= 28.28\angle -45°\Omega
\end{aligned}
$$

➢ b- The Impedance Triangle (Phasor Diagram)




$0{,}2\,mH$

$20j$

$V = 10\sin(10^5 t + 60)$

$w = 10^5$

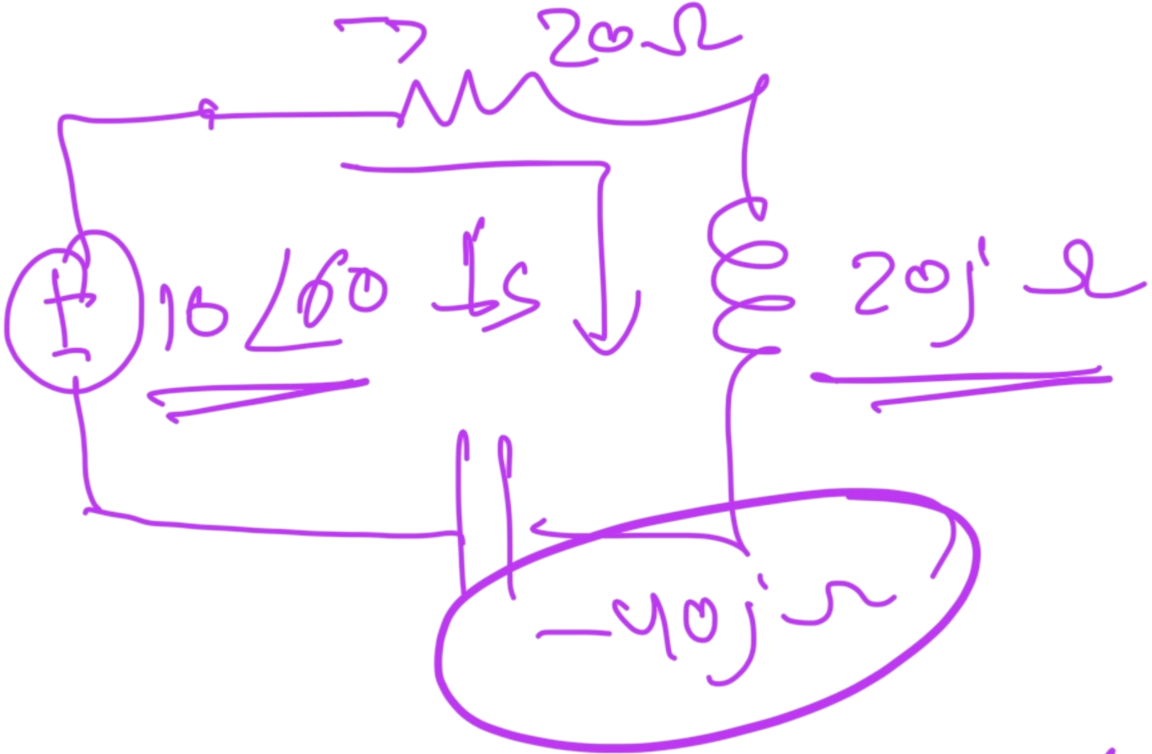
$0{,}25\,\mu F$ $\quad -40j$

$\dfrac{1}{0{,}25\times10^{-6}\times10^5\,j}$

$Z_R = R$

$Z_L = jwL$

$Z_C = \dfrac{1}{jwC}$




Series

$$V_s = I_s Z_T \Rightarrow I_s = \frac{V_s}{Z_T}$$

$$I_s = \frac{10\angle 60}{20 + 20j - 40j} = 0{,}354\angle 105 \text{ A}$$

$$V_R = IR = 0{,}354\angle 105 \times 20 = \underline{\underline{7{,}08\angle 105}}$$

$$V_L = I Z_L = 0{,}354\angle 105 \times 20j = \underline{\underline{7{,}08}}\angle -165$$

$$V_C = I Z_C = 0{,}354\angle 105 \times -j40 = 14{,}16\angle 15$$

Voltage divider series

$$V_R = \frac{20}{20+20j-40j} \times 10\angle 60 = 7{,}08\angle 105$$

$$V_L = \frac{20j}{20+20j-40j} \times 10\angle 60 = 7{,}08\angle -165$$

$$V_C = \frac{-40j}{20+20j-40j} \times 10\angle 60 = 14{,}16\angle 15$$

# C-TOTAL IMPEDANCE FOR AC CIRCUITS $-180 \le \theta \le 180$

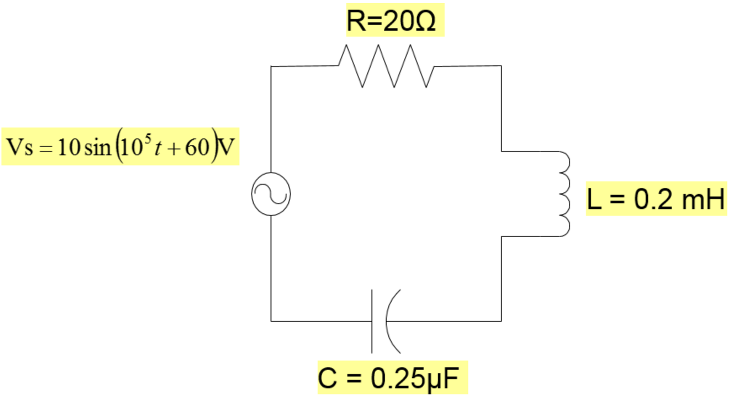
## 1-Example of series circuit:

a- Find Total Impedance

b-Draw Impedance Triangle (Phasor Diagram)

c- Find $i_s$, $v_R$, $v_C$, $v_L$

d- Find $v_R$, $v_C$, $v_L$ using Voltage Divider




✓ **Solution**

➢ c- $i_s$, $v_R$, $v_C$, $v_L$ :

d

$$i_s = \frac{v_s}{Z_T} = \frac{10\ \angle 60^\circ}{28.28\angle -45^\circ}$$

$$= 353\ \angle 105^\circ\ \ mA$$

$$v_s = i_s \cdot Z_R = (353\angle 105^\circ\ mA)\cdot(20\ \Omega) = 7\ \angle 105^\circ\ V$$

d

$$v_L = i_s \cdot Z_L = (353\angle 105^\circ\ mA)\cdot(20\angle 90^\circ\ \Omega) = 7\ \angle 195^\circ\ V$$

$$v_C = i_s \cdot Z_C = (353\angle 105^\circ\ mA)\cdot(40\angle -90^\circ\ \Omega) = 7\ \angle 15^\circ\ V$$

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# C-TOTAL IMPEDANCE FOR AC CIRCUITS

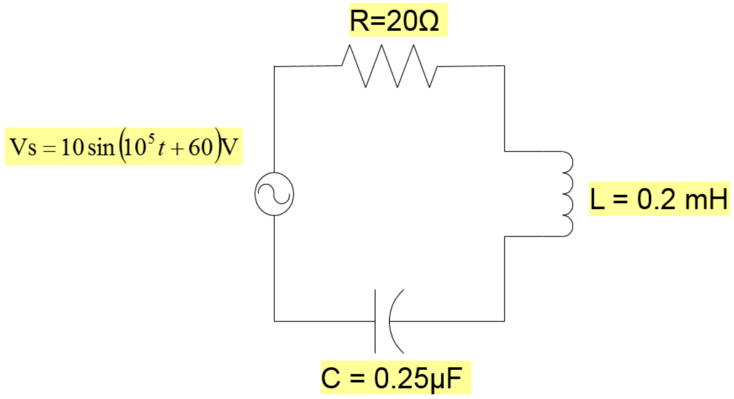
## 1-Example of series circuit:

a- Find Total Impedance

b-Draw Impedance Triangle (Phasor Diagram)

c- Find $i_s$, $v_R$, $v_C$, $v_L$

d- Find $v_R$, $v_C$, $v_L$ using Voltage Divider




✓ **Solution**

- d- Find $v_R$, $v_C$, $v_L$ using Voltage Divider
- Voltage divider still works, too.

$$v_R = \frac{Z_R}{Z_T} v_s = \frac{(20\Omega\angle 0^o)}{(28.28\ \angle -45^o)} . 10 . V\angle 60^o \ = 7\ \angle 105^o$$

$$v_L = \frac{Z_L}{Z_T} v_s = \frac{(20\Omega\angle 90^o)}{(28.28\ \angle -45^o)} . 10 . V\angle 60^o \ = 7\ \angle 195^o$$

$$v_C = \frac{Z_C}{Z_T} v_s = \frac{(20\Omega\angle -90^o)}{(28.28\ \angle -45^o)} . 10 . V\angle 60^o \ = 7\ \angle 15^o$$

# C-TOTAL IMPEDANCE FOR AC CIRCUITS

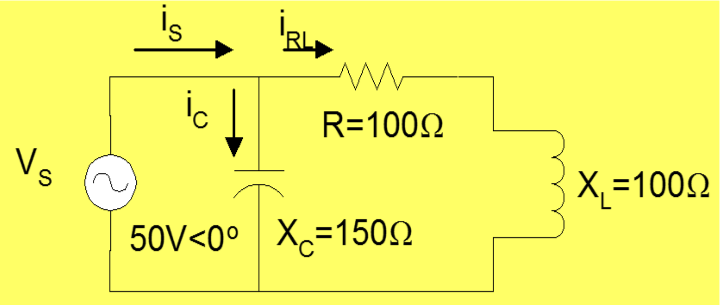
## 2-Example of Parallel circuit:

a- Find the total Impedance

b-Draw Impedance Triangle (Phasor Diagram)

c- Find $i_s$, $v_R$, $v_C$, $v_L$

d- Find $v_R$, $v_C$, $v_L$ using Voltage Divider

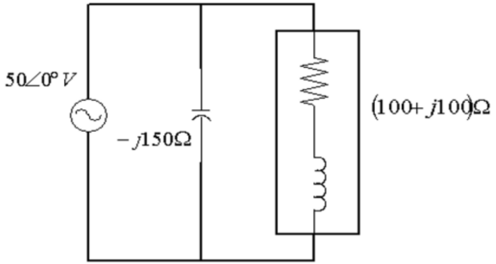



✓ **Solution** ➢ a- The total Impedance

➢ The circuit given is already in frequency domain:

$$Z_R = 100\Omega\angle 0^o \quad = 100\,\Omega$$
$$Z_C = 150\Omega\angle -90^o = -j150\,\Omega$$
$$Z_L = 100\Omega\angle 90^o \quad = j100\,\Omega$$

➢ **Circuit simplification**



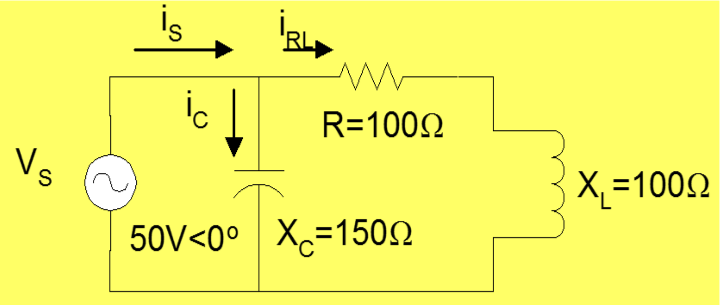
## 2-Example of Parallel circuit:

a- Find the total Impedance

b-Draw Impedance Triangle (Phasor Diagram)

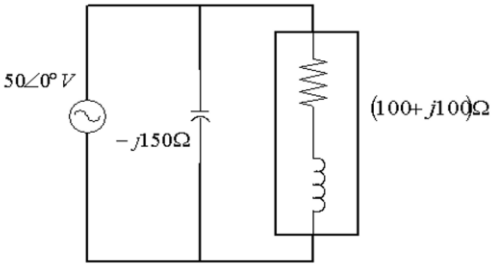
c- Find $i_s$, $v_R$, $v_C$, $v_L$

d- Find $v_R$, $v_C$, $v_L$ using Current Divider




✓ **Solution** ➢ a- The total Impedance

➢ The circuit given is already in frequency domain:




$$Z_{total} = (Z_C) \| (Z_R + Z_L)$$
$$= (150\angle -90^\circ) \| [(100) + (100\angle 90^\circ)]$$
$$= (150\angle -90^\circ) \| (100 + j100)$$
$$= (150\angle -90^\circ) \| (141.42\angle 45^\circ)$$
$$\therefore Z_{total} = [Y_{total}]^{-1}$$
$$= \left[\frac{1}{150\angle -90^\circ} + \frac{1}{141.42\angle 45^\circ}\right]^{-1}$$

$$\therefore Z_{total} = \boxed{\left(\frac{1}{150}(\cos(90^\circ) + j\sin(90^\circ)) + \frac{1}{144.42}(\cos(-45^\circ) + j\sin(-45^\circ))\right)^{-1}}$$

$$= (0.005 \quad +j0.0016)^{-1}$$

$$= \left(\left(\sqrt{0.005^2 + 0.0016^2}\right) < \tan^{-1}\left(\frac{0.0016}{0.005}\right)\right)^{-1}$$

$$= (0.00524 \quad < 17.744^\circ)^{-1} \simeq 190{<}{-18^\circ}$$

$$\therefore Z_{total} \simeq 180 - j60\,\Omega$$

# C-TOTAL IMPEDANCE FOR AC CIRCUITS

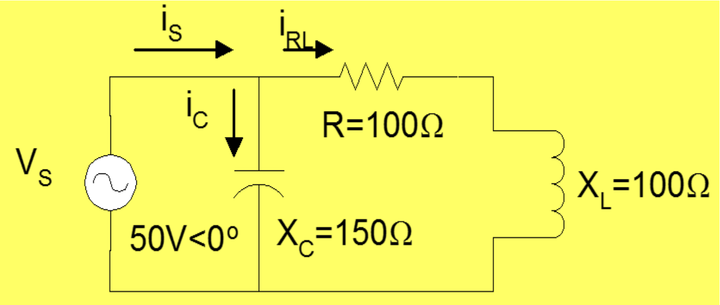
## 2-Example of Parallel circuit:

a- Find the total Impedance

b-Draw Impedance Triangle (Phasor Diagram)

c- Find $i_s$, $v_R$, $v_C$, $v_L$

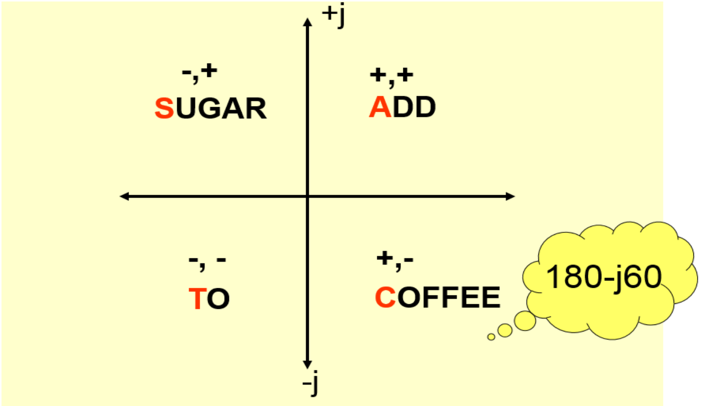
d- Find $v_R$, $v_C$, $v_L$ using Current Divider

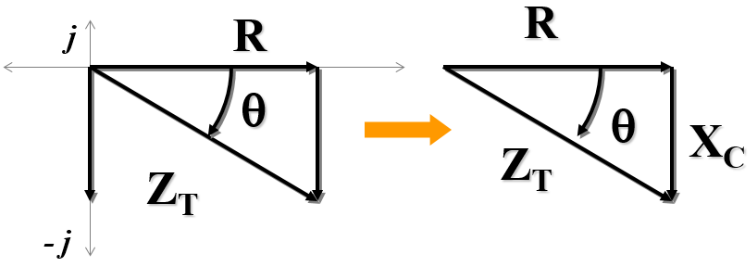



✓ **Solution**

➢ b- The Impedance Triangle (Phasor Diagram)

$$\therefore Z_{total} \simeq 180 - j60\,\Omega$$



# C-TOTAL IMPEDANCE FOR AC CIRCUITS

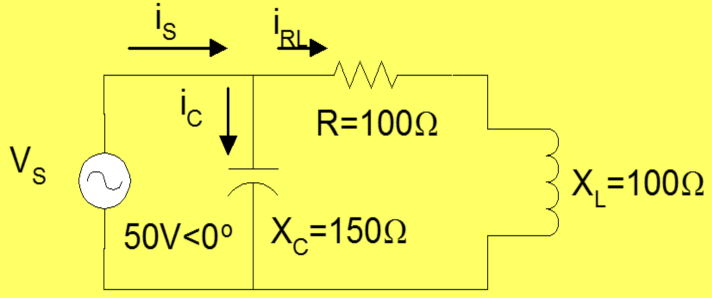
## 2-Example of Parallel circuit:

a- Find the total Impedance

b-Draw Impedance Triangle (Phasor Diagram)

c- Find $i_s$, $v_R$, $v_C$, $v_L$

d- Find $v_R$, $v_C$, $v_L$ using Current Divider




✓ **Solution**

➢ c- $i_s$, $v_R$, $v_C$, $v_L$ :

$$\therefore Z_{total} \simeq 180 - j60\,\Omega$$

$$i_s = \frac{v_s}{Z_T} = \frac{50\angle 0^\circ}{190\angle -18^\circ} = 263\angle 18^\circ\ mA$$

$$v_S = v_C = (v_R + v_L) = 50\angle 0^\circ\ V$$

$$\Rightarrow i_C = \frac{v_S}{Z_C} = \frac{50\angle 0^\circ}{150\angle -90^\circ} = 333\angle 90^\circ\ mA$$

$$\Rightarrow i_{RL} = \frac{v_S}{Z_R + Z_L} = \frac{50\angle 0^\circ}{100 + (100\angle -90^\circ)} = 353\angle -45^\circ\ mA$$

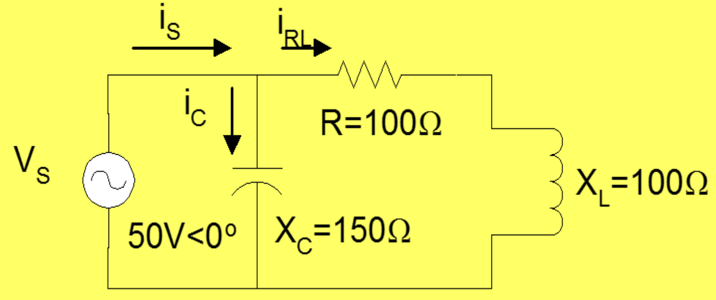
## 2-Example of Parallel circuit:

a- Find the total Impedance

b-Draw Impedance Triangle (Phasor Diagram)

c- Find $i_s$, $v_R$, $v_C$, $v_L$

d- Find $v_R$, $v_C$, $v_L$ using Current Divider




✓ **Solution**

➢ d- Find $v_R$, $v_C$, $v_L$ using Current Divider

$$i_c = \frac{Z_T}{Z_C} i_S = \frac{(190\angle -18°)}{(150\angle -90°)} \cdot (263\angle 18°\ mA)$$
$$= 333\angle 90°\ mA$$

$$i_{RL} = \frac{Z_T}{Z_R + Z_L} i_S = \frac{(190\angle -18°)}{(100)(100\angle 90°)} \cdot (263\angle 18°\ mA)$$
$$= 353\angle -45°\ mA$$

# D-Impedance as a Function of Frequency

| Element | Impedance |
|---|---|
| $L$ | $Z = j\omega L$ |
| $C$ | $Z = \dfrac{1}{j\omega C}$ |

- The Impedance Z of a circuit is a function of the frequency.

- Inductor is SHORT CIRCUIT at DC and OPEN CIRCUIT at high frequencies.
  Capacitor is OPEN CIRCUIT at DC and SHORT CIRCUIT at high frequencies.

$$Z_L = j\omega L$$

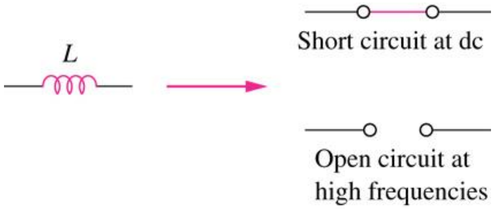
$$Z_L \to 0 \qquad \omega \to 0 \quad \text{(Short at DC)}$$

$$Z_L \to \infty \qquad \omega \to \infty \ (\text{Open as } \omega \to \infty)$$

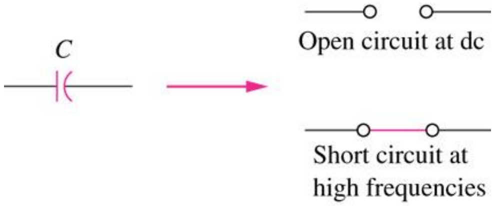
$$Z_C = \frac{1}{j\omega C}$$

$$Z_C \to \infty \qquad \omega \to 0 \quad \text{(Open at DC)}$$

$$Z_C \to 0 \qquad \omega \to \infty \quad (\text{Open as } \omega \to \infty)$$




(a)




(b)

# AC Analysis Techniques

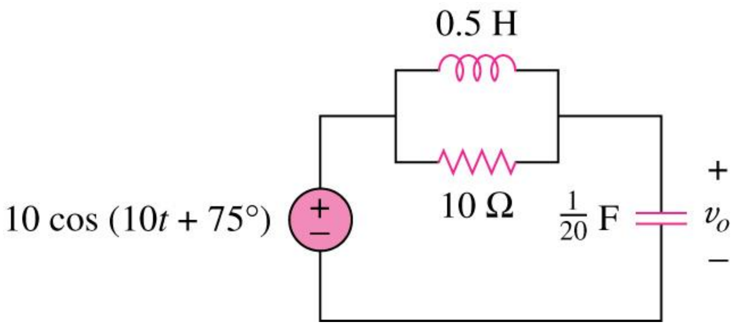
# AC Analysis Techniques

- Both KVL and KCL are hold in Phasor domain or more commonly called frequency domain.
- Moreover, the variables to be handled are phasors, which are complex numbers.
- All the mathematical operations involved are now in complex domain.

- **All DC circuit analysis principles apply to AC circuits.**
  - Voltage Division
  - Current Division
  - Circuit Reduction
  - Impedance equivalence
  - Y-Delta Transformation

# Example 22: Voltage Divider Rule

- Calculate the v0 in the given circuit




- Solution:

In the frequency domain,

the voltage source is $\mathbf{V}_s = 10\angle 75°$

the 0.5-H inductor is $j\omega L = j(10)(0.5) = j5$

the $\frac{1}{20}$-F capacitor is $\frac{1}{j\omega C} = \frac{1}{j(10)(1/20)} = -j2$

Let $\mathbf{Z}_1$ = impedance of the 0.5-H inductor in parallel with the 10-Ω resistor

and $\mathbf{Z}_2$ = impedance of the (1/20)-F capacitor

$$\mathbf{Z}_1 = 10 \,\|\, j5 = \frac{(10)(j5)}{10 + j5} = 2 + j4 \qquad \text{and} \qquad \mathbf{Z}_2 = -j2$$

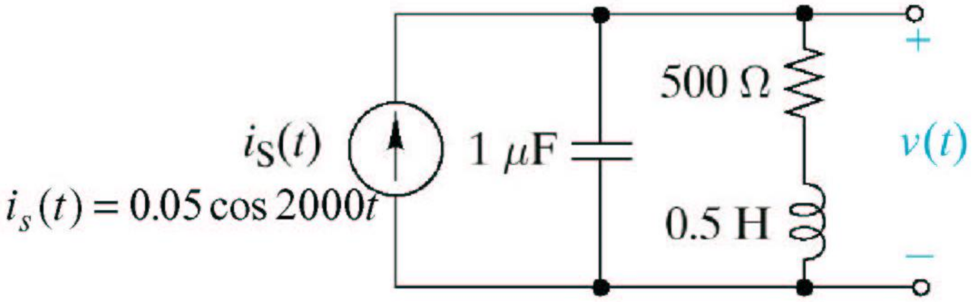
$$\mathbf{V}_o = \mathbf{Z}_2 / (\mathbf{Z}_1 + \mathbf{Z}_2)\,\mathbf{V}_s$$

$$\mathbf{V}_o = \frac{-j2}{2 + j4 - j2}(10\angle 75°) = \frac{-j(10\angle 75°)}{1 + j} = \frac{10\angle(75° - 90°)}{\sqrt{2}\angle 45°} = 7.071\angle\text{-}60°$$

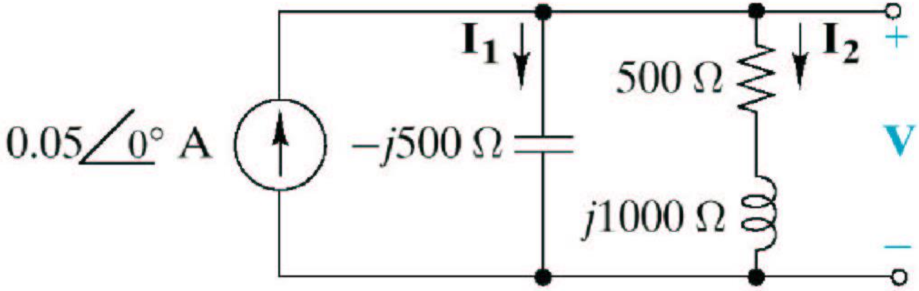
$$v_o(t) = \underline{\mathbf{7.071\cos(10t - 60°)\ V}}$$

# Example 23: Current Divider Rule

Find $I_1$ and $I_2$ in phasor domain, then convert them into time domain

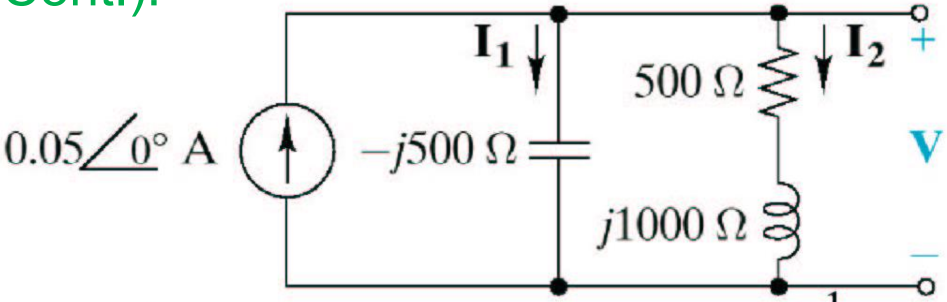



Solution:



# Example 23: Current Divider Rule

Solution (Cont.):




$$\hat{I}_1 = \frac{(R + j\omega L)}{R + j\omega L + \frac{1}{j\omega C}}\hat{I}$$

$$\hat{I}_1 = \frac{(500 + j1000)}{500 + j1000 - j500} 0.05\angle 0°$$

$$\hat{I}_2 = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}\hat{I}$$

$$\hat{I}_2 = \frac{-j500}{500 + j1000 - j500} 0.05\angle 0°$$

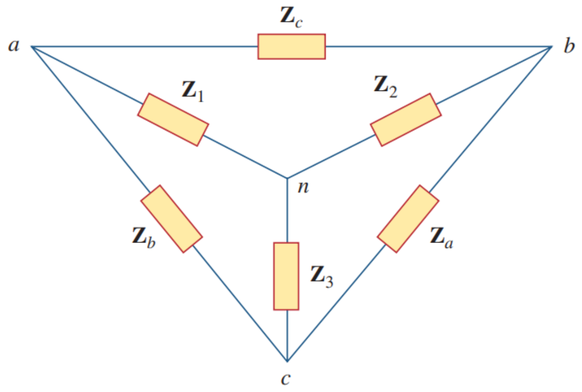
$$I_1 = 0.079\angle 18.4^o \text{ A}$$

$$I_2 = 0.03535\angle -135^o \text{ A}$$

$$i_1(t) = 79\cos(2000t + 18.4^o)\, mA$$

$$i_2(t) = 35.35\cos(2000t - 135^o)\, mA$$

# Delta-Wye ($\Delta - Y$) Transformation




*Y-Δ Conversion:*

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

*Δ-Y Conversion:*

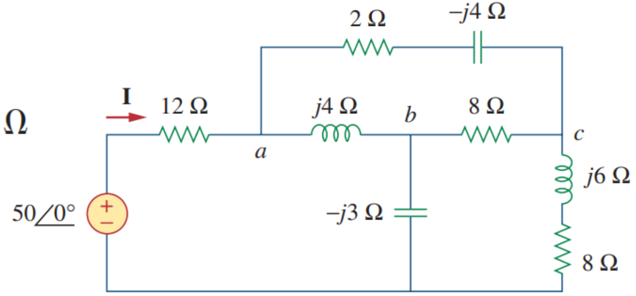
$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

# Example 24- ($\Delta - Y$) Transformation

Find current **I** in the circuit



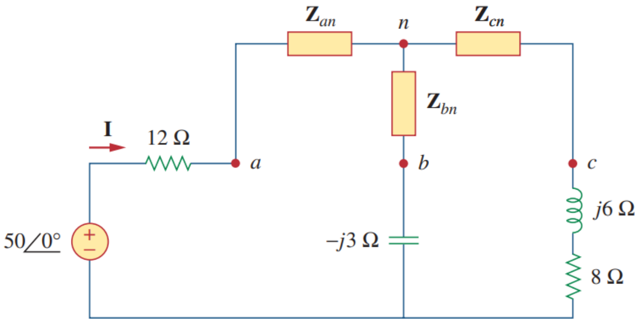

## Solution:

$$\mathbf{Z}_{an} = \frac{j4(2-j4)}{j4+2-j4+8} = \frac{4(4+j2)}{10} = (1.6+j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2-j4)}{10} = (1.6-j3.2)\ \Omega$$

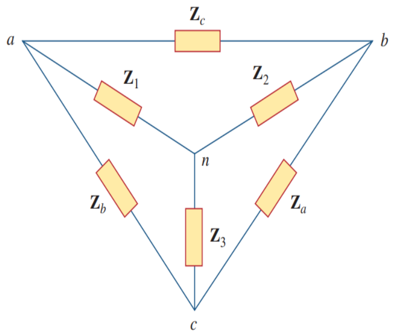
The total impedance at the source terminals is

$$\begin{aligned}\mathbf{Z} &= 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8)\\ &= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8)\\ &= 13.6 + j0.8 + \frac{j0.2(9.6+j2.8)}{9.6+j3}\\ &= 13.6 + j1 = 13.64\underline{/4.204^\circ}\ \Omega\end{aligned}$$




The desired current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\underline{/0^\circ}}{13.64\underline{/4.204^\circ}} = 3.666\underline{/-4.204^\circ}\ \text{A}$$




*Δ-Y Conversion:*

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

# Frequency Domain Analysis (Using Phasor)

Nodal Analysis

Mesh Analysis

Superposition,

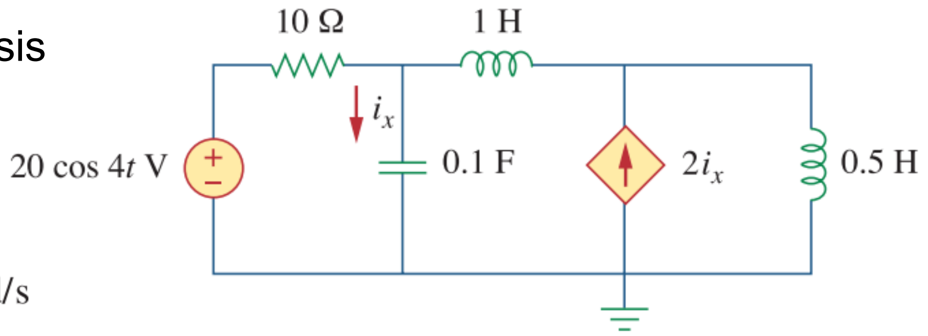
Source transformation

Thevenin theorem, and

Norton theorem

# Example 25- Nodal Analysis

Find $i_x$ in the following circuit using nodal analysis




## Solution

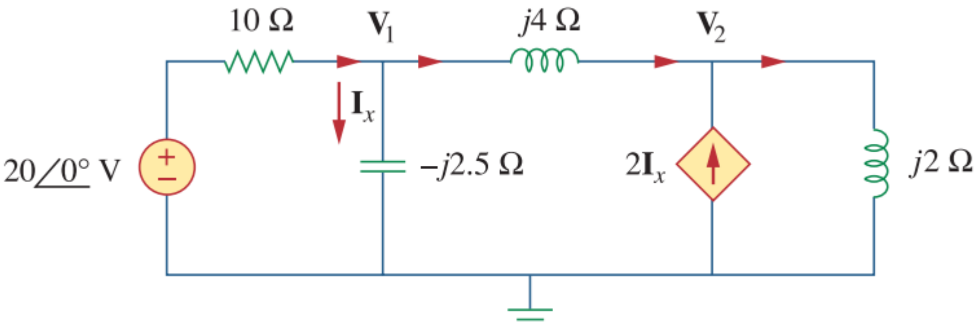
We first convert the circuit to the frequency domain:

$$20\cos 4t \quad \Rightarrow \quad 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s}$$

$$1\text{ H} \quad \Rightarrow \quad j\omega L = j4$$

$$0.5\text{ H} \quad \Rightarrow \quad j\omega L = j2$$

$$0.1\text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j2.5$$




Applying KCL at node 1,

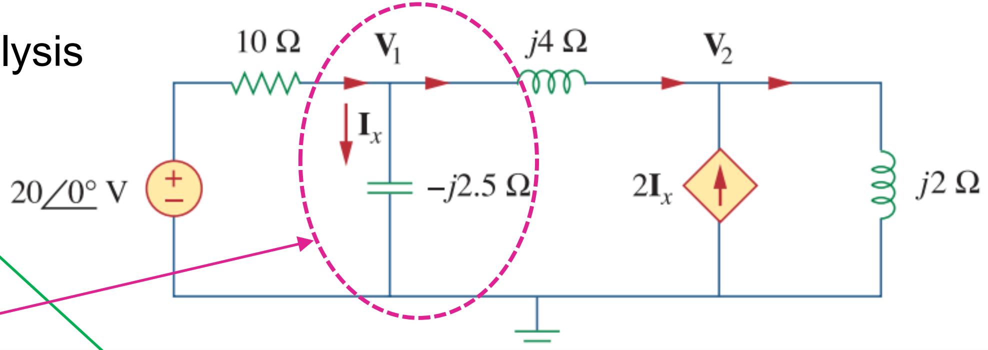
$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

or

$$\boxed{(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20}$$

# Example 25- Nodal Analysis

Find $i_x$ in the following circuit using nodal analysis




## Solution (continue)

$$\boxed{(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20}$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

But $\mathbf{I}_x = \mathbf{V}_1/-j2.5$. Substituting this gives

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

$$\boxed{11\mathbf{V}_1 + 15\mathbf{V}_2 = 0}$$

$$\boxed{\begin{bmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}}$$

$$\mathbf{V}_1 = 18.97\underline{/18.43^\circ}\text{ V}$$

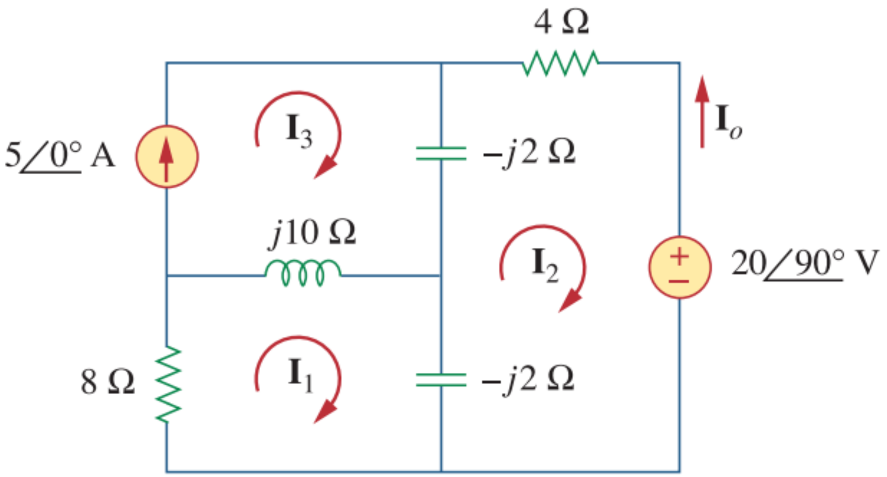
$$\mathbf{V}_2 = 13.91\underline{/198.3^\circ}\text{ V}$$

The current $\mathbf{I}_x$ is given by

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\underline{/18.43^\circ}}{2.5\underline{/-90^\circ}} = 7.59\underline{/108.4^\circ}\text{ A} \longrightarrow i_x = 7.59\cos(4t + 108.4^\circ)\text{ A}$$

# Example 26- Mesh Analysis

Find $I_o$ in the following circuit using mesh analysis




## Solution

Applying KVL to mesh 1, we obtain

- 8I1+j10(I1-I3)+-2j(I1-I2)=0

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0$$

For mesh 2,

- 20<90+ 4I2-j2(I2-I3) -j2(I2-I1) =0

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0$$

For mesh 3, $\mathbf{I}_3 = 5$

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

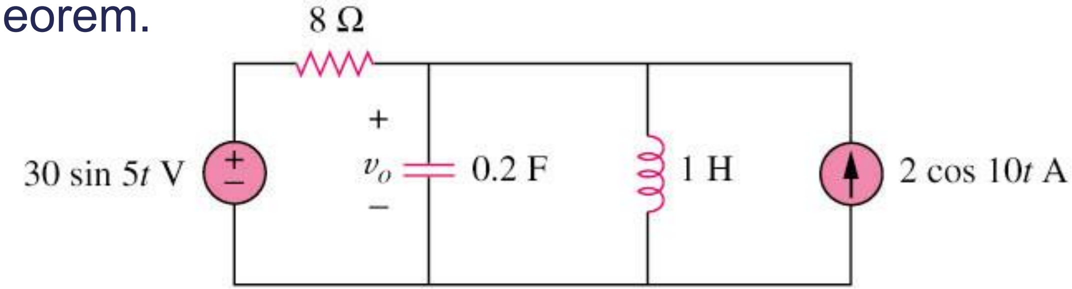
$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\angle -35.22^\circ}{68} = 6.12\angle -35.22^\circ \text{ A} \longrightarrow \boxed{\mathbf{I}_o = -\mathbf{I}_2 = 6.12\angle 144.78^\circ \text{ A}}$$

# Superposition Theorem

When a circuit has sources operating at different frequencies,

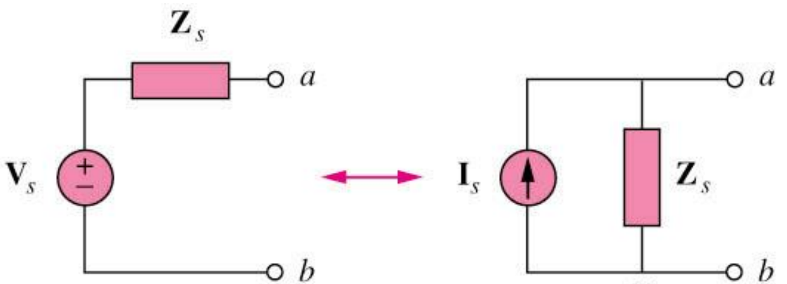
- The separate phasor circuit for each frequency must be solved independently, and
- The total response is the sum of time-domain responses of all the individual phasor circuits.

**Example** Calculate $v_o$ in the circuit of figure shown below using the superposition theorem.



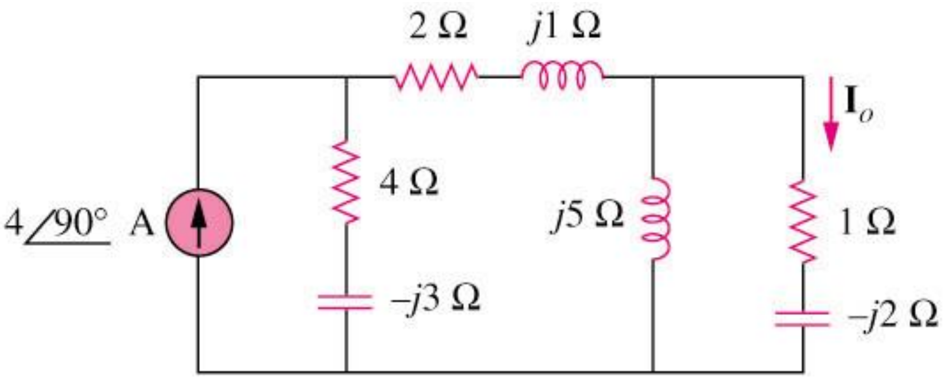

**Answer:** $V_o$ = **4.631 sin(5t – 81.12) + 1.051 cos(10t – 86.24) V**

# Source Transformation




$\mathbf{V}_s = \mathbf{Z}_s \mathbf{I}_s$ $\qquad$ $\mathbf{I}_s = \dfrac{\mathbf{V}_s}{\mathbf{Z}_s}$

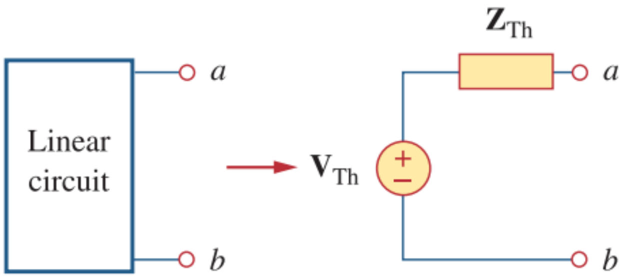
**Example** Find $Io$ in the circuit of figure below using the concept of source transformation.



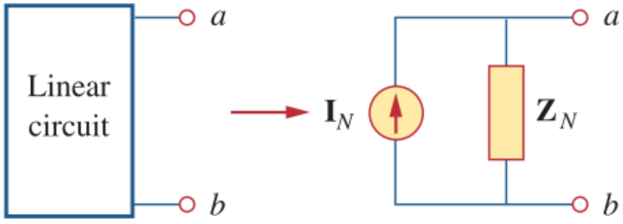

**Answer:** : $I_o$ = **3.288+j99.46 A**

# Thevenin and Norton Theorems in AC circuits

- Thevenin's and Norton theorems are applied to ac circuits in the same way as they are to DC circuits
- The ONLY additional effort is the need to manipulate complex numbers.




Thevenin equivalent.




Norton equivalent.

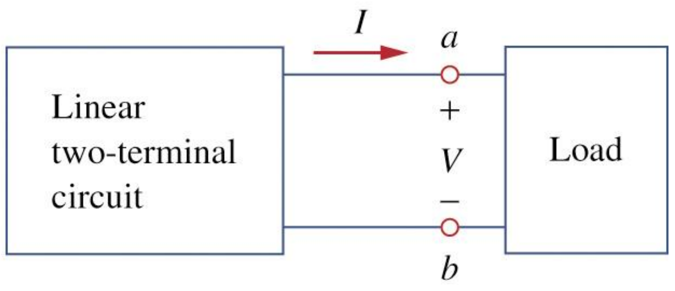
Where; $V_{Th} = V_{ab\text{-}OC}$ is the open circuit voltage (PHASOR) between terminals a-b,
$I_N = I_{ab\text{-}SC}$ is the short circuit current (PHASOR) through the terminals a-b, and
$Z_{Th}$ is the input or equivalent IMPEDANCE at the terminals when the independent source are turn off.

# How to Find Thevenin Equivalent Circuit in AC circuits

- First, **Open the circuit** (remove the load) at the **points** of interest **a-b**

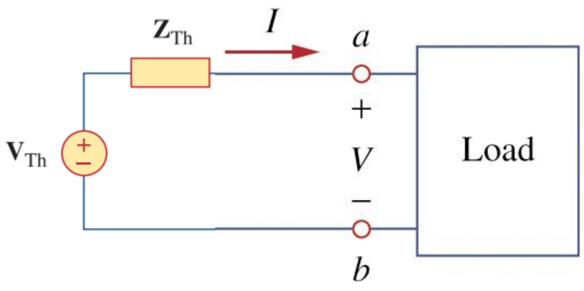
1- $V_{th}$= Open circuit voltage (keep all sources intact). **Note: Vth is Phasor**

2- $Z_{th}$= Open circuit **equivalent Impedance** appears at terminals **a-b** while all **independent sources=0** (voltage source=SC , current source=OC).




Original Circuit

≡




Thevenin equivalent circuit

# Example 10.8- Thevenin Equivalent Circuit

Obtain the Thevenin equivalent at terminals a-b of the following circuit.

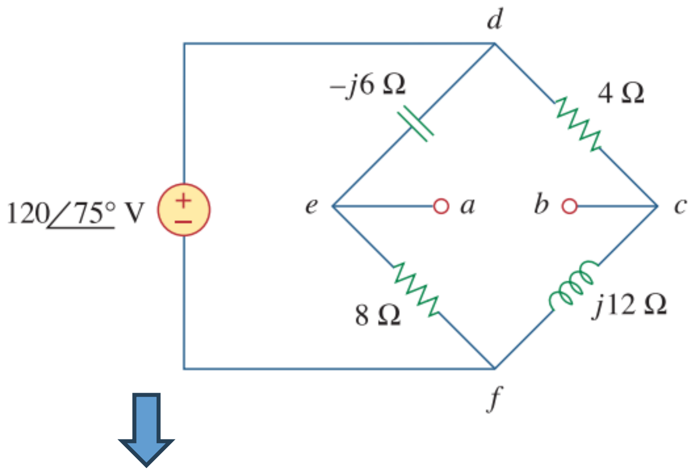
Solution

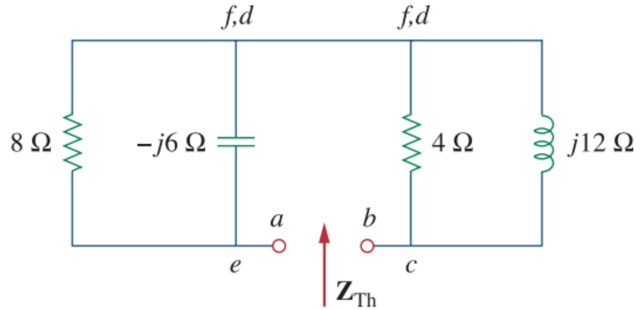
1- For Zth

$$\mathbf{Z}_1 = -j6 \parallel 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega$$

$$\mathbf{Z}_2 = 4 \parallel j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

$$\boxed{\mathbf{Z}_{\text{Th}} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega}$$



# Example 10.8- Thevenin Equivalent Circuit

Obtain the Thevenin equivalent at terminals a-b of the following circuit.

## Solution (continue)

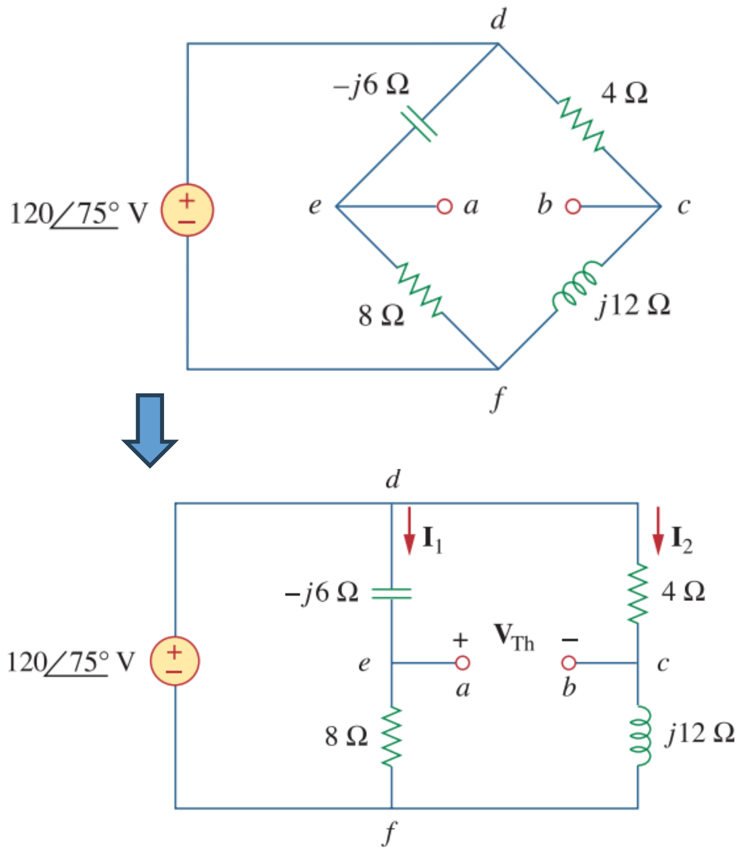
### 2- For Vth

$$\mathbf{I}_1 = \frac{120\angle 75^\circ}{8 - j6}\ \text{A}, \qquad \mathbf{I}_2 = \frac{120\angle 75^\circ}{4 + j12}\ \text{A}$$

Applying KVL around loop *bcdeab*

$$\mathbf{V}_{\text{Th}} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

$$\begin{aligned}\mathbf{V}_{\text{Th}} = 4\mathbf{I}_2 + j6\mathbf{I}_1 &= \frac{480\angle 75^\circ}{4 + j12} + \frac{720\angle 75^\circ + 90^\circ}{8 - j6} \\ &= 37.95\angle 3.43^\circ + 72\angle 201.87^\circ \\ &= -28.936 - j24.55 = 37.95\angle 220.31^\circ\ \text{V}\end{aligned}$$

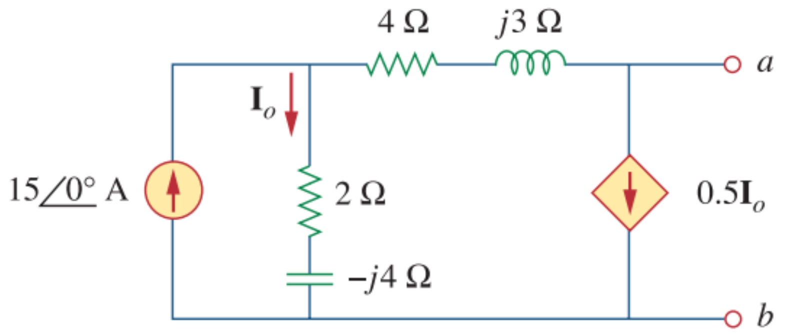


# Example 10.9- Thevenin Equivalent Circuit

Obtain the Thevenin equivalent at terminals a-b of the following circuit.

## Solution

### 1- For Vth




To find $\mathbf{V}_{Th}$, we apply KCL at node 1

$$15 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \Rightarrow \quad \mathbf{I}_o = 10 \text{ A}$$

Applying KVL to the loop on the right-hand side

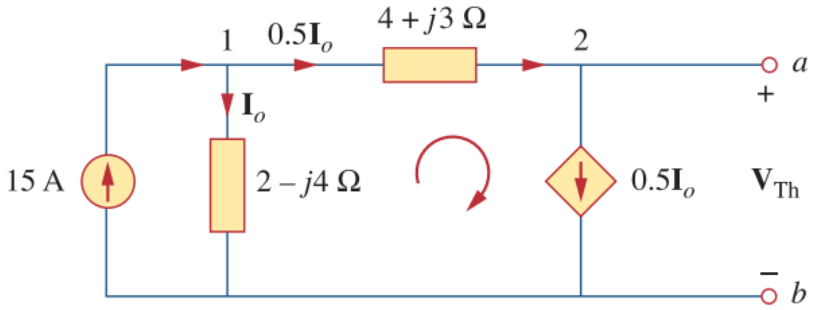
$$-\mathbf{I}_o(2 - j4) + 0.5\mathbf{I}_o(4 + j3) + \mathbf{V}_{Th} = 0$$

or

$$\mathbf{V}_{Th} = 10(2 - j4) - 5(4 + j3) = -j55$$

Thus, the Thevenin voltage is

$$\boxed{\mathbf{V}_{Th} = 55\angle -90^\circ \text{ V}}$$

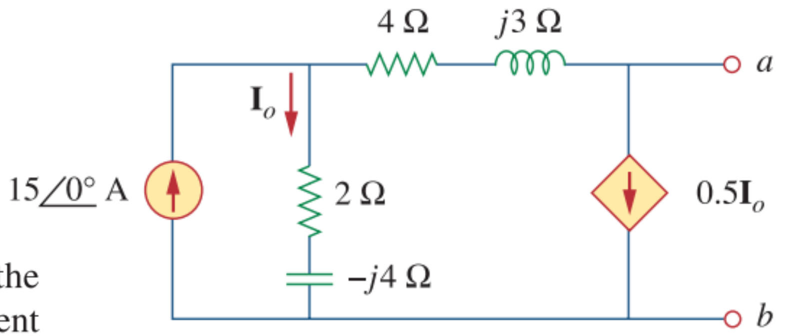


# Example 10.9- Thevenin Equivalent Circuit

Obtain the Thevenin equivalent at terminals a-b of the following circuit.

Solution (continue)

2- For Zth




To obtain $\mathbf{Z}_{\text{Th}}$, we remove the independent source. Due to the presence of the dependent current source, we connect a 3-A current source (3 is an arbitrary value chosen for convenience here, a number divisible by the sum of currents leaving the node) to terminals *a-b*

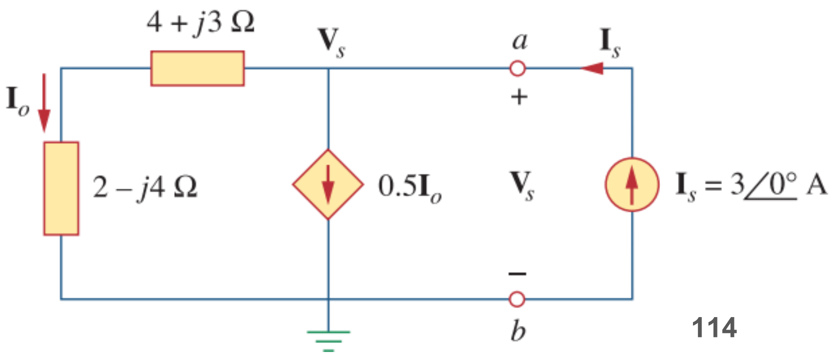
$$3 = \mathbf{I}_o + 0.5\mathbf{I}_o \qquad \Rightarrow \qquad \mathbf{I}_o = 2 \text{ A}$$

Applying KVL to the outer loop in Fig. 10.26(b) gives

$$\mathbf{V}_s = \mathbf{I}_o(4 + j3 + 2 - j4) = 2(6 - j)$$

The Thevenin impedance is

$$\boxed{\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6 - j)}{3} = 4 - j0.6667\ \Omega}$$



# Trigonometric Identities


- Sine and cosine form conversions.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

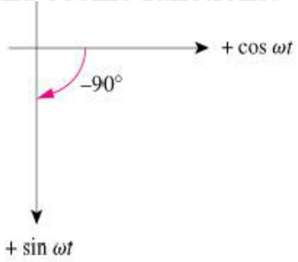
$$\sin(\omega t \pm 180°) = -\sin \omega t$$
$$\cos(\omega t \pm 180°) = -\cos \omega t$$
$$\sin(\omega t \pm 90°) = \pm \cos \omega t$$
$$\cos(\omega t \pm 90°) = \mp \sin \omega t$$

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta)$$

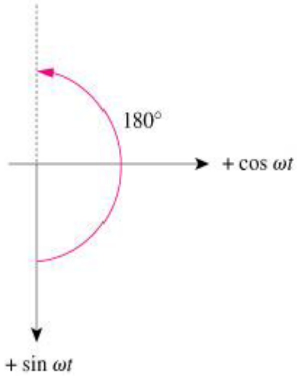
Where

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B}{A}$$

Graphically relating sine and cosine functions.




$$\cos(\omega t - 90°) = \sin \omega t$$




$$\sin(\omega t + 180°) = -\sin \omega t$$

# End of Lecture

Questions?