



SYLLABUS

Part I: COURSE INFORMATION

Semester	472	Academic Year	2025-2026
Department	General Studies	Specialization	Mathematics
Course Title	Integral Calculus		
Course Code	MATH 108	Prerequisite	MATH 107
Credit Hours	4	Weekly Contact Hours	4

Part II: TEXTBOOK

1.	Anton, H., Bivens, I., & Davis, S. (2013). <i>Calculus: Late Transcendentals</i> (10th ed.). Singapore: John Wiley & Sons Inc. ISBN: 978-1-11809248-4
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Part III: COURSE DESCRIPTION

This course designed to provide fundamental knowledge of integral calculus and its applications. It includes topics on definite and indefinite integration involving algebraic and transcendental functions in one variable, fundamental theorem of Calculus, various techniques of integration, average value and root mean square value of a function, areas between two curves, arc length of a curve, area of a surface of revolution, numerical integration, and differential equations of first-order with applications.

Part IV: COURSE LEARNING OUTCOME (CLO) – ASSESSMENT CHART

Upon completion of the course, the student should be able to:	Assessment *							
	Q1	Q2	HW1	ME	Q3	Q4	HW2	FE
CLO 1. analyze concepts and procedures related to integral calculus and first-order differential equations			5	2			5	4
CLO 2. evaluate integrals of basic functions by Riemann sums, fundamental theorem of calculus, and the substitution rule	7.5			10.5				6
CLO 3. apply definite integration to compute areas between curves, volumes of solids of revolution, lengths of plane curves, and areas of surfaces of revolution		5		5				2
CLO 4. evaluate integrals involving logarithmic functions, exponential functions, inverse trigonometric functions, and hyperbolic functions		2.5		2.5	2			5
CLO 5. perform advanced integration calculations and approximations					5.5	2		15
CLO 6. solve first-order ordinary differential equations and related applications						5.5		8
TOTAL	7.5	7.5	5	20	7.5	7.5	5	40

* Q = Quiz

HW = Homework

ME = Midterm Exam

FE = Final Exam (comprehensive)



Part V: ATTENDANCE

1. Check your EDUGATE account for the official class time, location, and absences.
2. Attendance will be checked at the start of each period as per ETA Rules and Regulations:
 - a. If you do not attend the class for any reason, you will be marked ABSENT for the corresponding missed periods.
 - b. If you arrive after the first 5 minutes, you will be marked ABSENT for that period.
 - c. If you arrive after the instructor called all students' names/IDs but during the first 5 minutes, you will be marked LATE for that period and will count as 1/3 of an absence.
 - d. If you were late three times, then these will be counted as 1 full absence.
3. In a semester, you will get a “DN” status if you exceed **20% absences**.
4. If you have a valid reason, and wanted your absence to be removed, contact the Office of Student Affairs and present your valid excuse there.

Part VI: TIPS ON EFFECTIVE STUDYING

- Schedule your **study time** and be **consistent** over the semester.

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
CLASS	Study Time		CLASS	Study Time		Study Time

- **Copy lecture notes** and **ask questions**.
- Apply the **Feynman technique** for learning new concepts:
 1. **Choose** a concept and organize it in your **own words**.
 2. **Teach** the concept to a beginner.
 3. **Fill the Gaps** that you may have missed.
 4. **Refine** your notes and explanation of the concept.
- **Test yourself** on the topics of the course.
- Do not simply memorize. **Studying is not memorizing**.

copy notes during the class and clarify your doubts

can be used before class or after class

organize a concept using simple language and write it on a paper

explain the concept to a beginner; identify weaknesses and issues in your explanation

in case you have trouble explaining, get feedback from your audience and research the concept again – go back to your resources or find new resources to fill the gaps

further simplify and refine your notes and explanations; repeat these steps as needed

solve the practice exercises and previous tests or exams

you MUST **apply your knowledge** and **develop mathematical skills** in this course

Part VII: Examples, Exercises, Homework, Office Hours

1. The **suggested examples** and **practice exercises** are the **MINIMUM** required level of **mathematical maturity** to be successful in this course.
2. Answer all the **practice exercises** without looking at any course-related materials.
3. Do your **homework** seriously, learn from it, and use it to extend your knowledge and skills in the course.
4. Utilize your instructor's **office hours** for help, as needed.



Part VIII: PACING SCHEDULE & PRACTICE EXERCISES

Week No.	Textbook's Section (§)	TOPIC	REQUIRED PRACTICE EXERCISES
1-4	4.1	An Overview of the Area Problem	§4.1: 1, 4, 5, 11, 14, 16, 17, 18, 22
	4.2	The Indefinite Integral	§4.2: 1,7, 10, 14, 18, 19, 24, 27, 30, 31, 34,39,64, 67, 68
	4.3	Integration by Substitution	§4.3: 2, 4,8, 12, 19,23, 24, 28, 32, 35, 47, 49
	4.4	The Definition of Area as a Limit	§4.4: 22, 23, 24, 26-30, 38, 42, 48,50
	4.5	The Definite Integral	§4.5: 14, 16, 18-20, 24, 25, 26
	4.6	The Fundamental Theorem of Calculus	§4.6: 11-13, 17, 19, 21, 23,26, 29, 44,49,53
	4.8	Average Value of a Function and Root Mean Square Value of a Function [p.341]	§4.8: 4, 5, 7, + definite integrals with continuous integrands over limits as closed intervals
	4.9	Evaluating Definite Integrals by Substitution	§4.9: 2, 6, 10, 17,20,21, 24, 29, 34
Quiz 1		Assessment for CLO 2	
5-7	5.1	Area Between Two Curves	§5.1: 1, 4, 5, 6, 7, 8, 9, 14, 29, 36
	5.2	Volumes by Slicing	§5.2: 1-9, 11, 18, 29, 30
	5.4	Length of a Plane Curve	§5.4: 3-8, 13, 14
	5.5	Area of a Surface of Revolution	§5.5: 1, 4, 5, 7
	6.2	Integrals Involving Logarithmic Functions	§6.2: 59, 62, 63, 65, 66, 69, 72
	6.3	Integrals Involving Exponential Functions	§6.3: 62, 64, 67, 68, 70,74-78
Quiz 2		Assessment for CLO 3 (66.67%) and CLO 4 (33.33%)	
8	Midterm Exam	Assessment for CLO 1 (10.00%), CLO 2 (52.50%), CLO 3 (25.00%), and CLO 4 (12.50%)	



Part VIII: PACING SCHEDULE & PRACTICE EXERCISES			
Week No.	Textbook's Section (§)	TOPIC	REQUIRED PRACTICE EXERCISES
9-12	6.7	Integrals Involving Inverse Trigonometric Functions	§6.7: 29-34, 37-39
	6.8	Integrals of Hyperbolic Functions	§6.8: 29, 30, 33, 34, 36
	7.2	Integration by Parts	§7.2: 2, 6, 8, 10, 14, 18, 22, 27, 29, 36, 47, 48, 49, 50, 59, 61, 62
	7.3	Integrating Trigonometric Functions	§7.3: 2, 3, 10, 29, 30, 68, 69, 70
	7.4	Trigonometric Substitutions	§7.4: 2, 3, 6, 12, 7, 10, 34
	7.5	Integrating Rational Functions by Partial Fractions	§7.5: 10, 13, 14, 23, 29
Quiz 3		Assessment for CLO 4 (26.67%) and CLO 5 (73.33%)	
13-15	7.7	Numerical Integration	§7.7: 1, 2, 3, 4, 5, 27, 30, 41, 43
	8.2	Separation of Variables	§8.2: 2, 3, 4, 10, 11, 12, 63
	8.4	First-Order Differential Equations and Applications	§8.4: 1, 2, 3, 4, 5, 8, 10, 23, 27
	8.2	Modeling with First-order Differential Equations	§8.2: 29, 30, 31, 32, 34, 35
	--	Bernoulli Differential Equation	Handout exercises
Quiz 4		Assessment for CLO 5 (26.67%) and CLO 6 (73.33%)	
16-17	Final Exam	Assessment for CLO 1 (10.00%), CLO 2 (15.00%), CLO 3 (5.00%), CLO 4 (12.50%), CLO 5 (37.50%), and CLO 6 (20.00%)	

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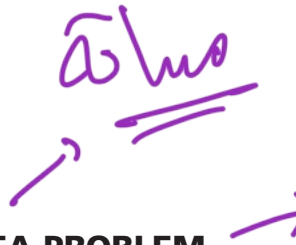


HelleM/Shutterstock

If a dragster moves with varying velocity over a certain time interval, it is possible to find the distance it travels during that time interval using techniques of calculus.

INTEGRATION

In this chapter we will introduce “integration,” a process motivated by the problem of computing the area of plane regions. After an informal overview of the problem, we will discuss a surprising relationship between integration and differentiation that is known as the Fundamental Theorem of Calculus. We will then apply integration to continue our study of rectilinear motion and to define the “average value” of a function. We conclude the chapter by studying some consequences of the chain rule in integral calculus.



4.1 AN OVERVIEW OF THE AREA PROBLEM

In this introductory section we will consider the problem of calculating areas of plane regions with curvilinear boundaries. All of the results in this section will be reexamined in more detail later in this chapter. Our purpose here is simply to introduce and motivate the fundamental concepts.

THE AREA PROBLEM

Formulas for the areas of polygons, such as squares, rectangles, triangles, and trapezoids, were well known in many early civilizations. However, the problem of finding formulas for regions with curved boundaries (a circle being the simplest example) caused difficulties for early mathematicians.

The first real progress in dealing with the general area problem was made by the Greek mathematician Archimedes, who obtained areas of regions bounded by circular arcs, parabolas, spirals, and various other curves using an ingenious procedure that was later called the *method of exhaustion*. The method, when applied to a circle, consists of inscribing a succession of regular polygons in the circle and allowing the number of sides to increase indefinitely. (See Figure 4.1.1 on the next page.) As the number of sides increases, the polygons tend to “exhaust” the region inside the circle, and the areas of the polygons become better and better approximations of the exact area of the circle.

To see how this works numerically, let $A(n)$ denote the area of a regular n -sided polygon inscribed in a circle of radius 1. Table 4.1.1 shows the values of $A(n)$ for various choices of n . Note that for large values of n the area $A(n)$ appears to be close to π (square units), as one would expect. This suggests that for a circle of radius 1, the method of exhaustion is equivalent to an equation of the form

$$\lim_{n \rightarrow \infty} A(n) = \pi$$

Since Greek mathematicians were suspicious of the concept of “infinity,” they avoided its use in mathematical arguments. As a result, computation of area using the method of exhaustion was a very cumbersome procedure. It remained for Newton and Leibniz to obtain a general method for finding areas that explicitly used the notion of a limit. We will discuss their method in the context of the following problem.

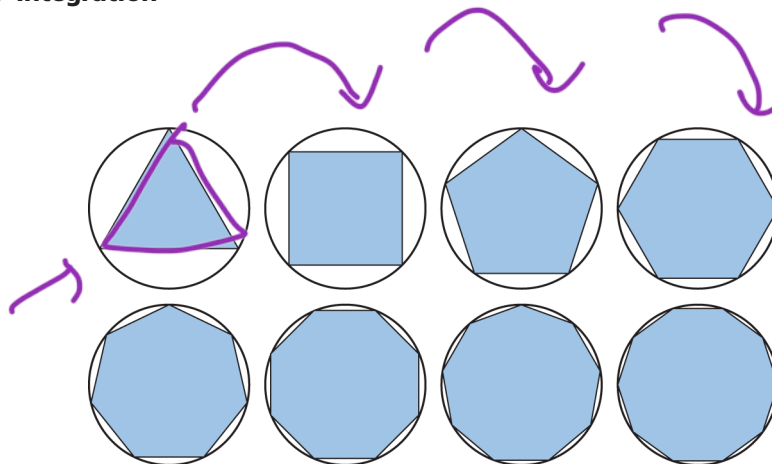
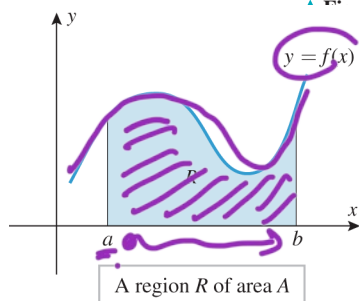


Table 4.1.1

n	$A(n)$
100	3.13952597647
200	3.14107590781
300	3.14136298250
400	3.14146346236
500	3.14150997084
1000	3.14157198278
2000	3.14158748588
3000	3.14159035683
4000	3.14159136166
5000	3.14159182676
10,000	3.14159244688

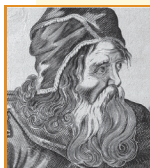
Handwritten purple annotations on the table: arrows pointing to the first three rows, underlines under the last three rows, and a large purple checkmark to the right of the final row.

Figure 4.1.1



4.1.1 THE AREA PROBLEM Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area A of the region R that lies between the graph of f and the interval $[a, b]$ on the x -axis (Figure 4.1.2).

Figure 4.1.2



Archimedes (287 B.C.–212 B.C.) Greek mathematician and scientist. Born in Syracuse, Sicily, Archimedes was the son of the astronomer Pheidias and possibly related to Heiron II, king of Syracuse. Most of the facts about his life come from the Roman biographer, Plutarch, who inserted a few tantalizing pages about

him in the massive biography of the Roman soldier, Marcellus. In the words of one writer, “the account of Archimedes is slipped like a tissue-thin shaving of ham in a bull-choking sandwich.”

Archimedes ranks with Newton and Gauss as one of the three greatest mathematicians who ever lived, and he is certainly the greatest mathematician of antiquity. His mathematical work is so modern in spirit and technique that it is barely distinguishable from that of a seventeenth-century mathematician, yet it was all done without benefit of algebra or a convenient number system. Among his mathematical achievements, Archimedes developed a general method (exhaustion) for finding areas and volumes, and he used the method to find areas bounded by parabolas and spirals and to find volumes of cylinders, paraboloids, and segments of spheres. He gave a procedure for approximating π and bounded its value between $3\frac{10}{71}$ and $3\frac{1}{7}$. In spite of the limitations of the Greek numbering system, he devised methods for finding square roots and invented a method based on the Greek myriad (10,000) for representing numbers as large as 1 followed by 80 million billion zeros.

Of all his mathematical work, Archimedes was most proud of his discovery of a method for finding the volume of a sphere—he showed that the volume of a sphere is two-thirds the volume of the smallest cylinder that can contain it. At his request, the figure of a sphere and cylinder was engraved on his tombstone.

In addition to mathematics, Archimedes worked extensively in mechanics and hydrostatics. Nearly every schoolchild knows Archimedes as the absent-minded scientist who, on realizing that a floating object displaces its weight of liquid, leaped from his bath and ran naked through the streets of Syracuse shouting, “Eureka, Eureka!”—(meaning, “I have found it!”). Archimedes actually created the discipline of hydrostatics and used it to find equilibrium

positions for various floating bodies. He laid down the fundamental postulates of mechanics, discovered the laws of levers, and calculated centers of gravity for various flat surfaces and solids. In the excitement of discovering the mathematical laws of the lever, he is said to have declared, “Give me a place to stand and I will move the earth.”

Although Archimedes was apparently more interested in pure mathematics than its applications, he was an engineering genius. During the second Punic war, when Syracuse was attacked by the Roman fleet under the command of Marcellus, it was reported by Plutarch that Archimedes’ military inventions held the fleet at bay for three years. He invented super catapults that showered the Romans with rocks weighing a quarter ton or more, and fearsome mechanical devices with iron “beaks and claws” that reached over the city walls, grasped the ships, and spun them against the rocks. After the first repulse, Marcellus called Archimedes a “geometrical Briareus (a hundred-armed mythological monster) who uses our ships like cups to ladle water from the sea.”

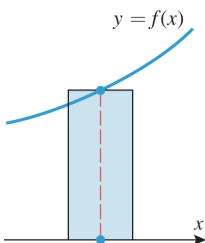
Eventually the Roman army was victorious and contrary to Marcellus’ specific orders the 75-year-old Archimedes was killed by a Roman soldier. According to one report of the incident, the soldier cast a shadow across the sand in which Archimedes was working on a mathematical problem. When the annoyed Archimedes yelled, “Don’t disturb my circles,” the soldier flew into a rage and cut the old man down.

Although there is no known likeness or statue of this great man, nine works of Archimedes have survived to the present day. Especially important is his treatise, *The Method of Mechanical Theorems*, which was part of a palimpsest found in Constantinople in 1906. In this treatise Archimedes explains how he made some of his discoveries, using reasoning that anticipated ideas of the integral calculus. Thought to be lost, the Archimedes palimpsest later resurfaced in 1998, when it was purchased by an anonymous private collector for two million dollars.

[Image: Archimedes Siracus. Line engraving by Remondini/Wellcome Library, London]

THE RECTANGLE METHOD FOR FINDING AREAS

One approach to the area problem is to use Archimedes' method of exhaustion in the following way:

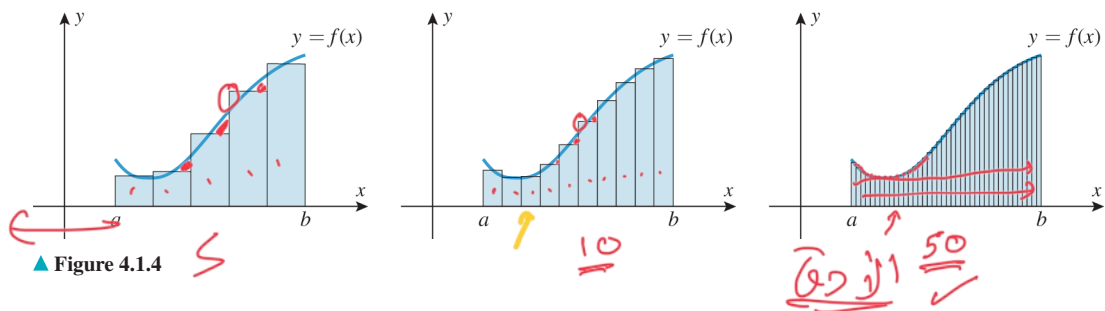


▲ Figure 4.1.3

- Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x -axis to any point on the curve $y = f(x)$ that is above the subinterval. In Figure 4.1.3 this point is above the center of the subinterval.
- For each n , the total area $A(n)$ of the rectangles can be viewed as an *approximation* to the area A . As n increases, these approximations will get better and better and will approach A as a limit

$$A = \lim_{n \rightarrow +\infty} A_n$$

(Figure 4.1.4). We will call this the *rectangle method* for computing A .



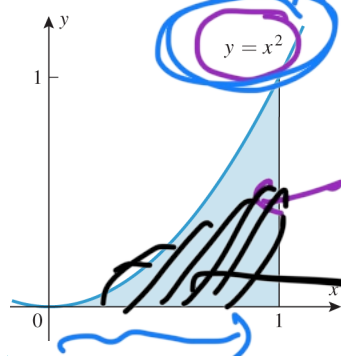
▲ Figure 4.1.4

ملاحظة

Logically speaking, we cannot really talk about computing areas without a precise mathematical definition of the term "area." Later in this chapter we will give such a definition, but for now we will treat the concept intuitively.

To illustrate this idea, we will use the rectangle method to approximate the area under the curve $y = x^2$ over the interval $[0, 1]$ (Figure 4.1.5). We will begin by dividing the interval $[0, 1]$ into n equal subintervals of length $1/n$; the endpoints of the subintervals occur at

$$0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}, 1$$



▲ Figure 4.1.5

(Figure 4.1.6). We will construct a rectangle over each of these subintervals whose height is the value of the function $f(x) = x^2$ at the right endpoint in the subinterval. The heights of our rectangles will be

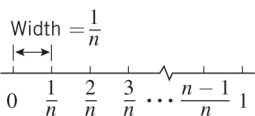
$$\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \dots, 1^2$$

and since each rectangle has a base of width $1/n$, the total area A_n of the n rectangles will be

$$A_n = \left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \dots + 1^2 \right] \left(\frac{1}{n}\right) \tag{1}$$

For example, if $n = 4$, then the total area of the four approximating rectangles would be

$$A_4 = \left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2 \right] \left(\frac{1}{4}\right) = \frac{15}{32} = 0.46875$$

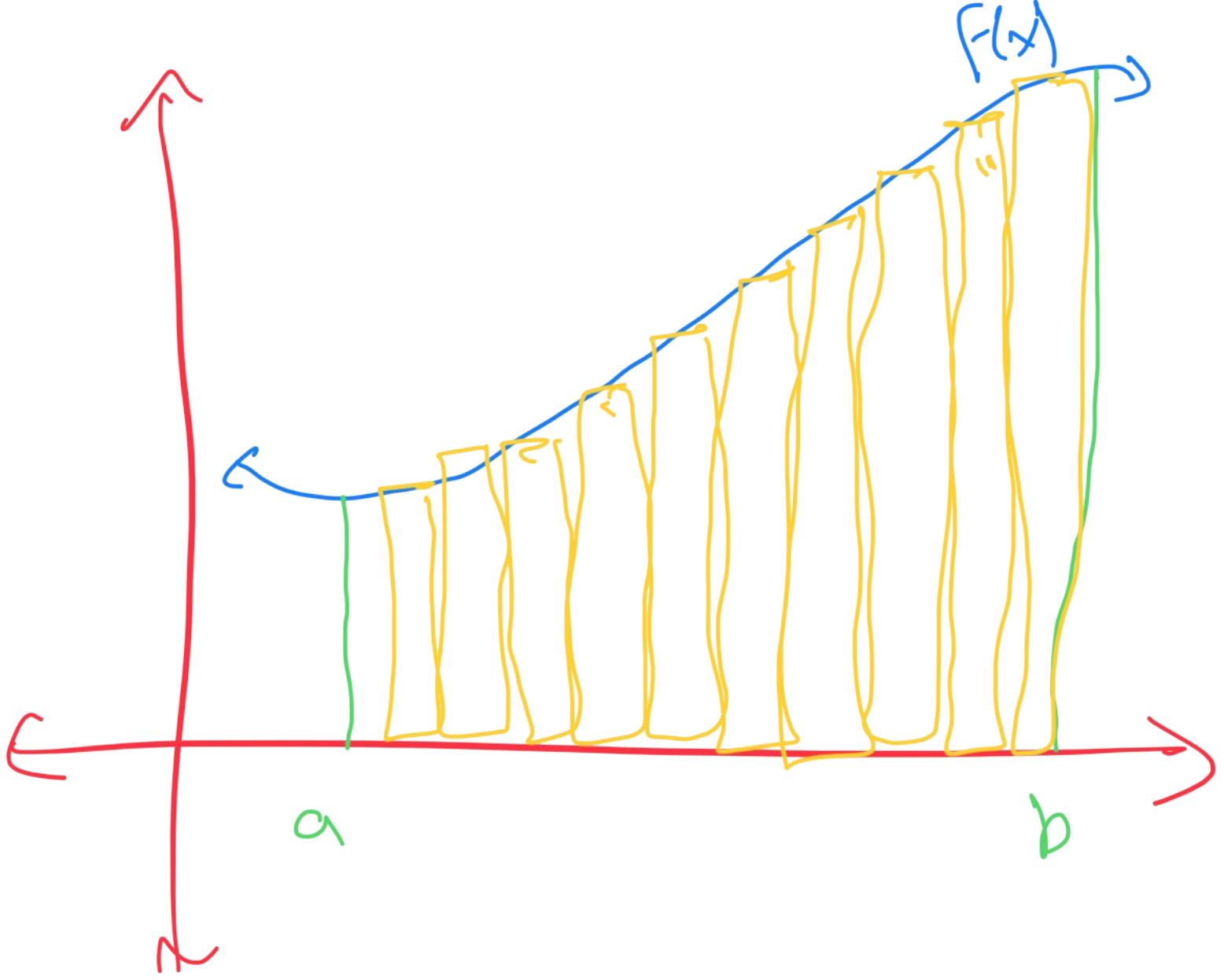


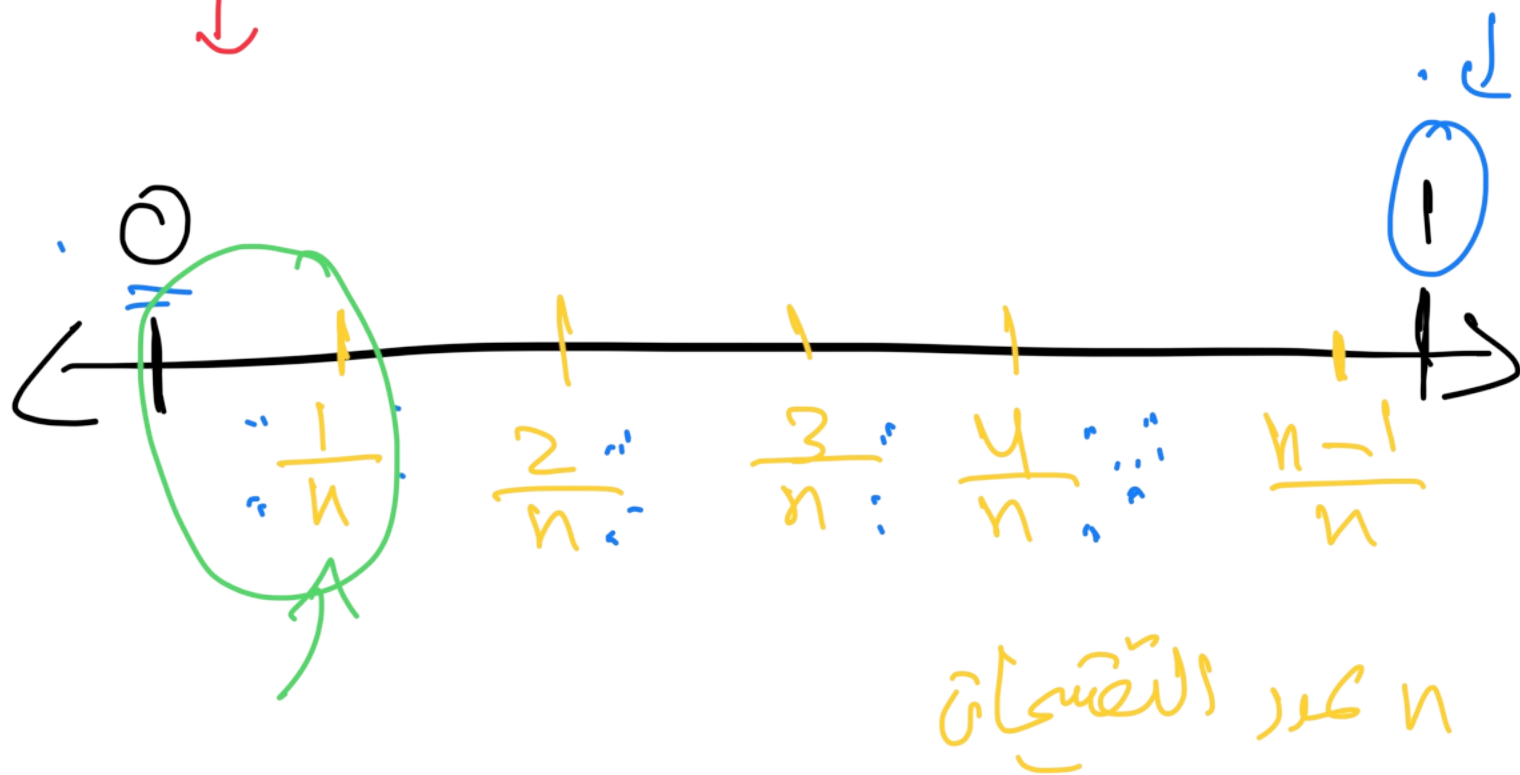
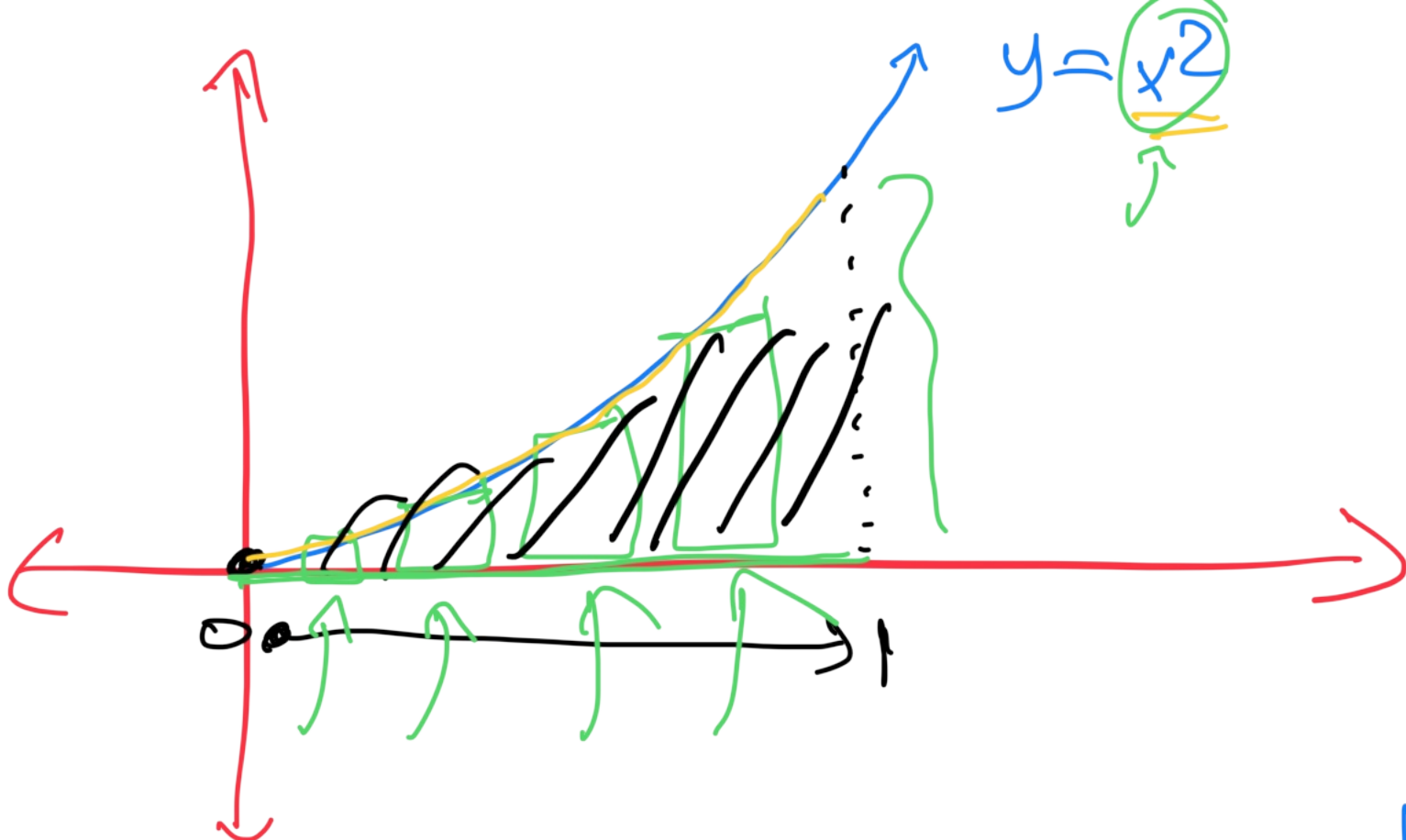
Subdivision of $[0, 1]$ into n subintervals of equal length

▲ Figure 4.1.6

Table 4.1.2 shows the result of evaluating (1) on a computer for some increasingly large values of n . These computations suggest that the exact area is close to $\frac{1}{3}$. Later in this chapter we will prove that this area is exactly $\frac{1}{3}$ by showing that

Handwritten notes in Arabic and yellow circles: $\lim_{n \rightarrow \infty} A_n = \frac{1}{3}$. Includes a diagram of a trapezoid and the formula $\frac{1}{2} \times \text{base} \times \text{height}$.





$n = 2$: $0 \rightarrow \frac{1}{2} \rightarrow 1$

$n = 5$: $0 \rightarrow \frac{1}{5} \rightarrow \frac{2}{5} \rightarrow \frac{3}{5} \rightarrow \frac{4}{5} \rightarrow 1$

$$h = u : 0 \rightarrow \frac{1}{4} \rightarrow \frac{2}{4} \rightarrow \frac{3}{4} \rightarrow 1$$

Area of strips } = الفول \times الارتفاع

$$A_M = \left[0 + \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2 \right] \times \frac{1}{4}$$

$$\frac{15}{8} \times \frac{1}{4}$$

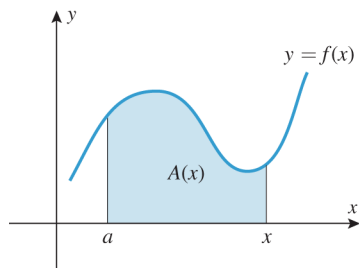
$$\underline{A_M} = 0,46875$$

TECHNOLOGY MASTERY

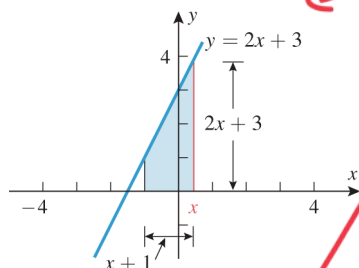
Use a calculating utility to compute the value of A_{10} in Table 4.1.2. Some calculating utilities have special commands for computing sums such as that in (1) for any specified value of n . If your utility has this feature, use it to compute A_{100} as well.

Table 4.1.2

n	4	10	100	1000	10,000	100,000
A_n	0.468750	0.385000	0.338350	0.333834	0.333383	0.333333



▲ Figure 4.1.7



▲ Figure 4.1.8

THE ANTIDERIVATIVE METHOD FOR FINDING AREAS

Although the rectangle method is appealing intuitively, the limits that result can only be evaluated in certain cases. For this reason, progress on the area problem remained at a rudimentary level until the latter part of the seventeenth century when Isaac Newton and Gottfried Leibniz independently discovered a fundamental relationship between areas and derivatives. Briefly stated, they showed that if f is a nonnegative continuous function on the interval $[a, b]$, and if $A(x)$ denotes the area under the graph of f over the interval $[a, x]$, where x is any point in the interval $[a, b]$ (Figure 4.1.7), then

$$A'(x) = f(x) \tag{2}$$

The following example confirms Formula (2) in a simple case.

► **Example 1** Find the area $A(x)$ between the graph of $f(x) = 2x + 3$ and the interval $[a, x] = [-1, x]$ and find the derivative $A'(x)$ of this area function.

Solution. Recall that the formula for the area of a trapezoid is $A = \frac{1}{2}(b + b')h$, where b and b' denote the lengths of the parallel sides of the trapezoid, and the altitude h denotes the distance between the parallel sides. From Figure 4.1.8 we see that

$$A(x) = \frac{1}{2}((2x + 3) + 1)(x - (-1)) = x^2 + 3x + 2$$

is the area of a trapezoid with parallel sides of lengths 1 and $2x + 3$ and with altitude $x - (-1) = x + 1$. For this area function,

$$A'(x) = 2x + 3 = f(x) \blacktriangleleft$$

Formula (2) is important because it relates the area function A and the region-bounding function f . Although a formula for $A(x)$ may be difficult to obtain directly, its derivative, $f(x)$, is given. If a formula for $A(x)$ can be recovered from the given formula for $A'(x)$, then the area under the graph of f over the interval $[a, b]$ can be obtained by computing $A(b)$.

The process of finding a function from its derivative is called **antidifferentiation**, and a procedure for finding areas via antidifferentiation is called the **antiderivative method**. To illustrate this method, let us revisit the problem of finding the area in Figure 4.1.5.

► **Example 2** Use the antiderivative method to find the area under the graph of $y = x^2$ over the interval $[0, 1]$.

Solution. Let x be any point in the interval $[0, 1]$, and let $A(x)$ denote the area under the graph of $f(x) = x^2$ over the interval $[0, x]$. It follows from (2) that

$$A'(x) = x^2 \tag{3}$$

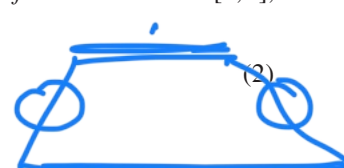
To find $A(x)$ we must look for a function whose derivative is x^2 . By guessing, we see that one such function is $\frac{1}{3}x^3$, so by Theorem 4.8.3

$$A(x) = \frac{1}{3}x^3 + C \tag{4}$$

for some real constant C . We can determine the specific value for C by considering the case where $x = 0$. In this case (4) implies that

$$A(0) = C \tag{5}$$

How does the solution to Example 2 change if the interval $[0, 1]$ is replaced by the interval $[-1, 1]$?



تشبه متحرقي

Ex ① Find $A(x)$ ✓

$$f(x) = 2x + 3 \quad \text{on} \quad [-1, x]$$

$$\underline{\underline{A'(x) = f(x) = 2x + 3}}$$

derivative, which is the solution

Area \rightarrow

derivative

A' \rightarrow

derivative zero

$$\underline{\underline{A' = 2x + 3}}$$

$$\frac{d}{dx}(A) = \underline{A'} = \underline{2x + 3}$$

anti derivative $A' = \underline{A}$

$$A(x) = \underline{x^2 + 3x} - 1$$

$$\frac{d}{dx}(x^2) \longrightarrow 2x$$

$$\frac{d}{dx}(3x) \longrightarrow \underline{3}$$

$$A(x) = (x^2 + 3x) - ((-1)^2 + (3(-1)))$$

$$x^2 + 3x - (1 - 3) = \underline{\underline{x^2 + 3x + 2}}$$

Ex 2 anti derivative
ant, ant

Ex $\frac{d}{dx} (x^3) \rightarrow 3x^2$

anti derivative $3x^2 \rightarrow x^3$

Integral $\int 3x^2 \rightarrow x^3$

Ex $\frac{d}{dx} (\sin x) \rightarrow \cos x$

anti derivative $\cos x \rightarrow \sin x$

integral $\int \cos x = \sin x$

$\int x^2$

$$y = x^2$$

$\int [0, 1]$

$$A'(x) =$$

$$x^2$$

Antiderivate = $A(x)$
"un" "ant", "un" "ant"

$$A(x) = 3x^2$$

$$\frac{d}{dx} \left(\frac{1}{3} x^{3 \cdot -1} \right) \rightarrow x^2$$

$$\frac{1}{3} [3x^2] = x^2$$

$$A(x) = \frac{x^3}{3}$$

$$\left(\frac{4}{3}\right)^3 - \left(\frac{0}{3}\right)^3 = 0$$

$$A = \frac{1}{3} \Rightarrow \underline{\underline{0,3333\dots}}$$

$$\frac{d}{dx} (\text{Area}) \Rightarrow f(x)$$

$$\text{Area} = \int f(x) dx$$

= anti derivative $f(x)$



But if $x = 0$, then the interval $[0, x]$ reduces to a single point. If we agree that the area above a single point should be taken as zero, then $A(0) = 0$ and (5) implies that $C = 0$. Thus, it follows from (4) that

$$A(x) = \frac{1}{3}x^3$$

is the area function we are seeking. This implies that the area under the graph of $y = x^2$ over the interval $[0, 1]$ is

$$A(1) = \frac{1}{3}(1^3) = \frac{1}{3}$$

This is consistent with the result that we previously obtained numerically. ◀

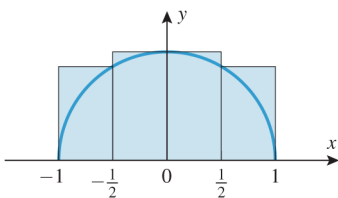
As Example 2 illustrates, antidifferentiation is a process in which one tries to “undo” a differentiation. One of the objectives in this chapter is to develop efficient antidifferentiation procedures.

THE RECTANGLE METHOD AND THE ANTIDERIVATIVE METHOD COMPARED

The rectangle method and the antiderivative method provide two very different approaches to the area problem, each of which is important. The antiderivative method is usually the more efficient way to *compute* areas, but it is the rectangle method that is used to formally *define* the notion of area, thereby allowing us to prove mathematical results about areas. The underlying idea of the rectangle approach is also important because it can be adapted readily to such diverse problems as finding the volume of a solid, the length of a curve, the mass of an object, and the work done in pumping water out of a tank, to name a few.

✓ **QUICK CHECK EXERCISES 4.1** (See page 208 for answers.)

1. Let R denote the region below the graph of $f(x) = \sqrt{1 - x^2}$ and above the interval $[-1, 1]$.
 - (a) Use a geometric argument to find the area of R .
 - (b) What estimate results if the area of R is approximated by the total area within the rectangles of the accompanying figure?



◀ Figure Ex-1

2. Suppose that when the area A between the graph of a function $y = f(x)$ and an interval $[a, b]$ is approximated by the areas of n rectangles, the total area of the rectangles is $A_n = 2 + (2/n)$, $n = 1, 2, \dots$. Then, $A = \underline{\hspace{2cm}}$.
3. The area under the graph of $y = x^2$ over the interval $[0, 3]$ is $\underline{\hspace{2cm}}$.
4. Find a formula for the area $A(x)$ between the graph of the function $f(x) = x$ and the interval $[0, x]$, and verify that $A'(x) = f(x)$.
5. The area under the graph of $y = f(x)$ over the interval $[0, x]$ is $A(x) = x + \sin x$. It follows that $f(x) = \underline{\hspace{2cm}}$.

EXERCISE SET 4.1

1–12 Estimate the area between the graph of the function f and the interval $[a, b]$. Use an approximation scheme with n rectangles similar to our treatment of $f(x) = x^2$ in this section. If your calculating utility will perform automatic summations, estimate the specified area using $n = 10, 50,$ and 100 rectangles. Otherwise, estimate this area using $n = 2, 5,$ and 10 rectangles. ■

1. $f(x) = \sqrt{x}$; $[a, b] = [0, 1]$

2. $f(x) = \frac{1}{x+1}$; $[a, b] = [0, 1]$

3. $f(x) = \sin x$; $[a, b] = [0, \pi]$

4. $f(x) = \cos x$; $[a, b] = [0, \pi/2]$

5. $f(x) = \frac{1}{x}$; $[a, b] = [1, 2]$

6. $f(x) = \cos x$; $[a, b] = [-\pi/2, \pi/2]$

7. $f(x) = \sqrt{1 - x^2}$; $[a, b] = [0, 1]$

8. $f(x) = \sqrt{1 - x^2}$; $[a, b] = [-1, 1]$

ii) $\sqrt{x} \leftarrow [0, 1]$



المساحة؟

$n = 2$: $0 \rightarrow \frac{1}{2} \rightarrow 1$

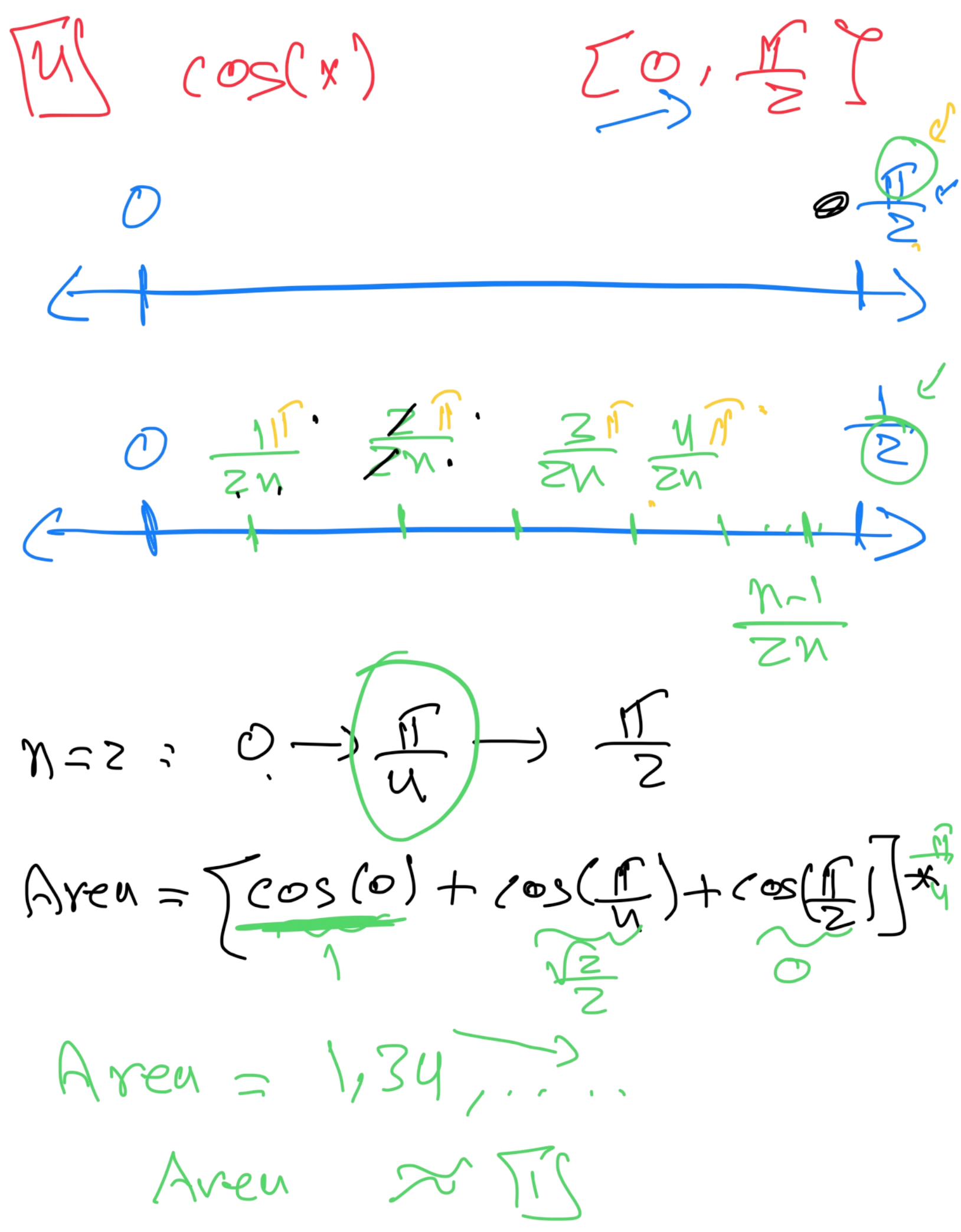
Area = $[\sqrt{0} + \sqrt{\frac{1}{2}} + \sqrt{1}] * \frac{1}{2} = 0.8535$

$n = 5$: $0 \rightarrow \frac{1}{5} \rightarrow \frac{2}{5} \rightarrow \frac{3}{5} \rightarrow \frac{4}{5} \rightarrow 1$

Area = $[\sqrt{0} + \sqrt{\frac{1}{5}} + \sqrt{\frac{2}{5}} + \sqrt{\frac{3}{5}} + \sqrt{\frac{4}{5}} + \sqrt{1}] * \frac{1}{5}$

Area = 0.7497

≈ 0.8
 0.6



9–14 Graph each function over the specified interval. Then use simple area formulas from geometry to find the area function $A(x)$ that gives the area between the graph of the specified function f and the interval $[a, x]$. Confirm that $A'(x) = f(x)$ in every case. ■

9. $f(x) = 3$; $[a, x] = [1, x]$

10. $f(x) = 5$; $[a, x] = [2, x]$

11. $f(x) = 2x + 2$; $[a, x] = [0, x]$

12. $f(x) = 3x - 3$; $[a, x] = [1, x]$

13. $f(x) = 2x + 2$; $[a, x] = [1, x]$

14. $f(x) = 3x - 3$; $[a, x] = [2, x]$

15–18 True–False Determine whether the statement is true or false. Explain your answer. ■

15. If $A(n)$ denotes the area of a regular n -sided polygon inscribed in a circle of radius 2, then $\lim_{n \rightarrow +\infty} A(n) = 2\pi$.

16. If the area under the curve $y = x^2$ over an interval is approximated by the total area of a collection of rectangles, the approximation will be too large.

17. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A'(x) = f(x)$.

18. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A(x)$ will be a continuous function.

FOCUS ON CONCEPTS

19. Explain how to use the formula for $A(x)$ found in the solution to Example 2 to determine the area between the graph of $y = x^2$ and the interval $[3, 6]$.

20. Repeat Exercise 19 for the interval $[-3, 9]$.

21. Let A denote the area between the graph of $f(x) = \sqrt{x}$ and the interval $[0, 1]$, and let B denote the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$. Explain geometrically why $A + B = 1$.

22. Let A denote the area between the graph of $f(x) = 1/x$ and the interval $[1, 2]$, and let B denote the area between the graph of f and the interval $[\frac{1}{2}, 1]$. Explain geometrically why $A = B$.

23–24 The area $A(x)$ under the graph of f and over the interval $[a, x]$ is given. Find the function f and the value of a . ■

23. $A(x) = x^2 - 4$

24. $A(x) = x^2 - x$

25. **Writing** Compare and contrast the rectangle method and the antiderivative method.

26. **Writing** Suppose that f is a nonnegative continuous function on an interval $[a, b]$ and that $g(x) = f(x) + C$, where C is a positive constant. What will be the area of the region between the graphs of f and g ?

✓ **QUICK CHECK ANSWERS 4.1** 1. (a) $\frac{\pi}{2}$ (b) $1 + \frac{\sqrt{3}}{2}$ 2. 2 3. 9 4. $A(x) = \frac{x^2}{2}$; $A'(x) = \frac{2x}{2} = x = f(x)$ 5. $\cos x + 1$

4.2 THE INDEFINITE INTEGRAL

In the last section we saw how antidifferentiation could be used to find exact areas. In this section we will develop some fundamental results about antidifferentiation.

ANTIDERIVATIVES

4.2.1 DEFINITION A function F is called an **antiderivative** of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx} \left[\frac{1}{3}x^3 \right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx} \left[\frac{1}{3}x^3 + C \right] = x^2 + 0 = f(x)$$