Since v and i are *in phase,* the element is a *resistor,* and

$$R = \frac{V_m}{I_m} = \frac{50\ \text{V}}{5\ \text{A}} = \mathbf{10\ \Omega}$$

14.3 FREQUENCY RESPONSE OF THE BASIC ELEMENTS

Thus far, each description has been for a set frequency, resulting in a fixed level of impedance for each of the basic elements. We must now investigate how a change in frequency affects the impedance level of the basic elements. It is an important consideration because most signals other than those provided by a power plant contain a variety of frequency levels. The last section made it quite clear that the reactance of an inductor or a capacitor is sensitive to the applied frequency. However, the question is, How will these reactance levels change if we steadily increase the frequency from a very low level to a much higher level?

Although we would like to think of every element as ideal, it is important to realize that every commercial element available today *will not respond in an ideal fashion for the full range of possible frequencies.* That is, each element is such that for a particular range of frequencies, it performs in an essentially ideal manner. However, there is always a range of frequencies in which the performance varies from the ideal. Fortunately, the designer is aware of these limitations and will take them into account in the design.

The discussion begins with a look at the response of the *ideal elements*—a response that will be assumed for the remaining chapters of this text and one that can be assumed for any initial investigation of a network. This discussion is followed by a look at the factors that cause an element to deviate from an ideal response as frequency levels become too low or high.

Ideal Response

Resistor *R* For an ideal resistor, you can assume that *frequency will have absolutely no effect on the impedance level,* as shown by the response in Fig. 14.16. Note that at 5 kHz or 20 kHz, the resistance of the resistor remains at 22 Ω; there is no change whatsoever. For the rest of the analyses in this text, the resistance level remains as the nameplate value, no matter what frequency is applied. This is not true for commercially available resistors with some more sensitive to the applied frequency than others, but for this text we will assume the resistors are frequency insensitive.

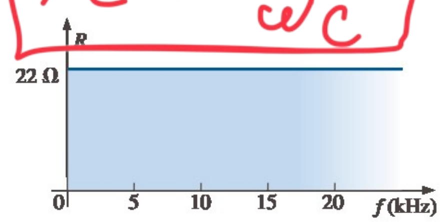


FIG. 14.16
R versus f for the range of interest.

Inductor *L* For the ideal inductor, the equation for the reactance can be written as follows to isolate the frequency term in the equation. The result is a constant times the frequency variable that changes as we move down the horizontal axis of a plot:

$$X_L = \omega L = 2\pi f L = (2\pi L)f = kf \qquad \text{with } k = 2\pi L$$

The resulting equation can be compared directly with the equation for a straight line:

$$y = mx + b = kf + 0 = kf$$

Inductor $X_L = \omega L$ $\omega = 2\pi f$

* $\omega = 0 \longrightarrow f = 0 \text{ Hz}$

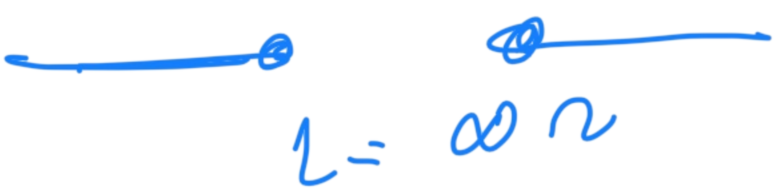
$\Longrightarrow$ DC source $f = 0$

$X_L = 0\,\Omega \Longrightarrow$ short circuit

=

* $\omega = \infty \longrightarrow f = \infty \text{ Hz}$

$X_L = \infty\,\Omega \Longrightarrow$ open circuit



Capacitor $X_C = \frac{1}{\omega C}$ $\qquad \omega = 2\pi f$

$\omega = 0 \implies f = 0 \text{ Hz}$

$\implies$ DC source $f = 0$

$X_C = \frac{1}{0} = \infty\ \Omega \implies$ open circuit

$X_C = \infty\ \Omega$

$\omega = \infty \implies f = \infty \text{ Hz}$

$X_C = \frac{1}{\infty} = 0\ \Omega \implies$ short circuit

$X_C = 0\ \Omega$

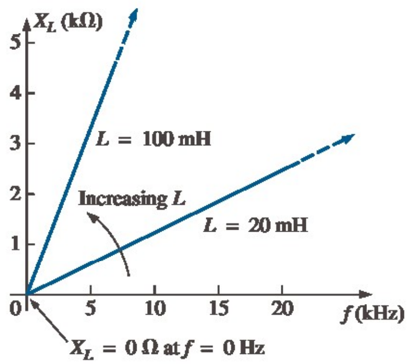


FIG. 14.17
X_L versus frequency.

where $b = 0$ and the slope is k or $2\pi L$. X_L is the y variable, and f is the x variable, as shown in Fig. 14.17. Since the inductance determines the slope of the curve, the higher the inductance, the steeper is the straight-line plot, as shown in Fig. 14.17 for two levels of inductance.

In particular, note that at $f = 0$ Hz, the reactance of each plot is zero ohms, as determined by substituting $f = 0$ Hz into the basic equation for the reactance of an inductor:

$$X_L = 2\pi fL = 2\pi(0\text{ Hz})L = 0\ \Omega$$

Since a reactance of zero ohms corresponds with the characteristics of a short circuit, we can conclude that

at a frequency of 0 Hz, an inductor takes on the characteristics of a short circuit, as shown in Fig. 14.18.

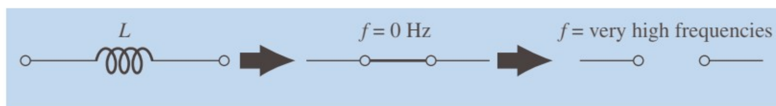


FIG. 14.18
Effect of low and high frequencies on the circuit model of an inductor.

As shown in Fig. 14.18, as the frequency increases, the reactance increases, until it reaches an extremely high level at very high frequencies. The result is that

at very high frequencies, the characteristics of an inductor approach those of an open circuit, as shown in Fig. 14.18.

The inductor, therefore, is capable of handling impedance levels that cover the entire range, from zero ohms to infinite ohms, changing at a *steady rate* determined by the inductance level. The higher the inductance, the faster it approaches the open-circuit equivalent.

Capacitor *C* For the capacitor, the equation for the reactance

$$X_C = \frac{1}{2\pi fC}$$

can be written as

$$X_C f = \frac{1}{2\pi C} = k \qquad \text{(a constant)}$$

which matches the basic format for a hyberbola:

$$yx = k$$

where X_C is the y variable, f the x variable, and k a constant equal to $1/(2\pi C)$.

Hyberbolas have the shape appearing in Fig. 14.19 for two levels of capacitance. Note that the higher the capacitance, the closer the curve approaches the vertical and horizontal axes at low and high frequencies.

At or near 0 Hz, the reactance of any capacitor is extremely high, as determined by the basic equation for capacitance:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(0\text{ Hz})C} \Rightarrow \infty\ \Omega$$

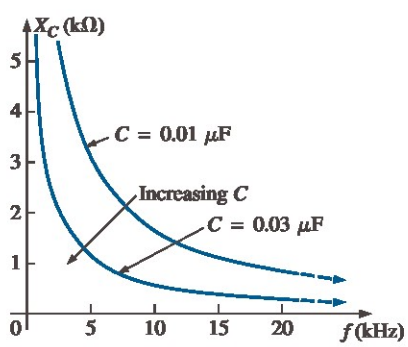


FIG. 14.19
X_C versus frequency.

The result is that

at or near 0 Hz, the characteristics of a capacitor approach those of an open circuit, as shown in Fig. 14.20.

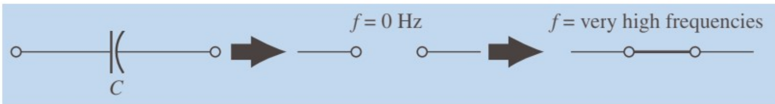


FIG. 14.20
Effect of low and high frequencies on the circuit model of a capacitor.

As the frequency increases, the reactance approaches a value of zero ohms. The result is that

at very high frequencies, a capacitor takes on the characteristics of a short circuit, as shown in Fig. 14.20.

It is important to note in Fig. 14.19 that the reactance drops very rapidly as the frequency increases. It is not a gradual drop as encountered for the rise in inductive reactance. In addition, the reactance sits at a fairly low level for a broad range of frequencies. In general, therefore, recognize that for capacitive elements, the change in reactance level can be dramatic with a relatively small change in frequency level.

Finally, recognize the following:

As frequency increases, the reactance of an inductive element increases, while that of a capacitor decreases, with one approaching an open-circuit equivalent as the other approaches a short-circuit equivalent.

Practical Response

Resistor *R* In the manufacturing process, every resistive element inherits some stray capacitance levels and lead inductances. For most applications, the levels are so low that their effects can be ignored. However, as the frequency extends beyond a few megahertz, it may be necessary to be aware of their effects. For instance, a number of carbon composition resistors have the frequency response appearing in Fig. 14.21.

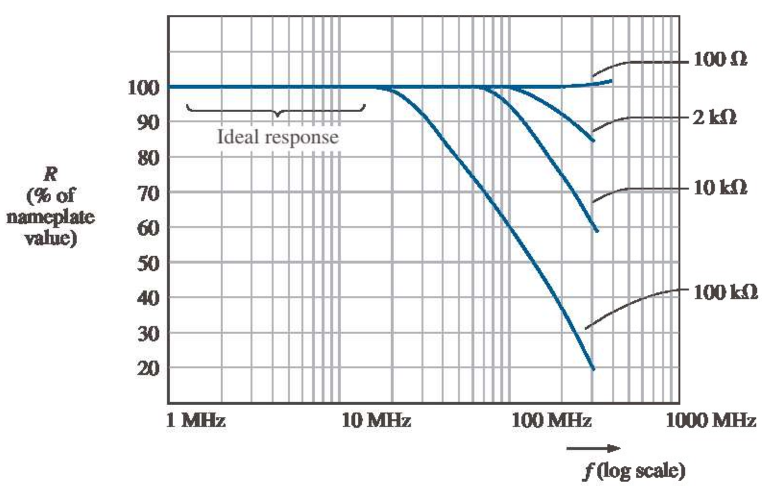


FIG. 14.21
Typical resistance-versus-frequency curves for carbon composition resistors.

The 100 Ω resistor is essentially stable up to about 300 MHz, whereas the 100 kΩ resistor starts to drop off at about 15 MHz. In general, therefore, this type of carbon composition resistor has the ideal characteristics of Fig. 14.16 for frequencies up to about 15 MHz. For frequencies of 100 Hz, 1 kHz, 150 kHz, and so on, the resistor can be considered ideal.

The horizontal scale of Fig. 14.21 is a log scale that starts at 1 MHz rather than zero as applied to the vertical scale. Logarithms are discussed in detail in Chapter 22, which describes why the scale cannot start at zero and the fact that the major intervals are separated by powers of 10. For now, simply note that log scales permit the display of a range of frequencies not possible with a linear scale such as was used for the vertical scale of Fig. 14.21. Imagine trying to draw a linear scale from 1 MHz to 1000 MHz using a linear scale. It would be an impossible task unless the horizontal length of the plot was enormous. As indicated above, a great deal more will be said about log scales in Chapter 22.

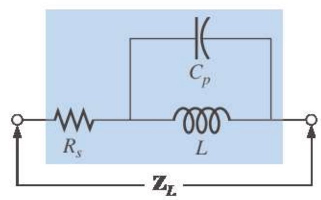


FIG. 14.22
Practical equivalent for an inductor.

Inductor *L* In reality, inductance can be affected by frequency, temperature, and current. A true equivalent for an inductor appears in Fig. 14.22. The series resistance R_s represents the copper losses (resistance of the many turns of thin copper wire); the eddy current losses (losses due to small circular currents in the core when an ac voltage is applied); and the hysteresis losses (losses due to core losses created by the rapidly reversing field in the core). The capacitance C_p is the stray capacitance that exists between the windings of the inductor.

For most inductors, the construction is usually such that the larger the inductance, the lower is the frequency at which the parasitic elements become important. That is, for inductors in the millihenry range (which is very typical), frequencies approaching 100 kHz can have an effect on the ideal characteristics of the element. For inductors in the microhenry range, a frequency of 1 MHz may introduce negative effects. This is not to suggest that the inductors lose their effect at these frequencies but rather that they can no longer be considered ideal (purely inductive elements).

Fig. 14.23 is a plot of the magnitude of the reactance X_L of Fig. 14.22 versus frequency. Note that up to about 2 MHz, the impedance increases almost linearly with frequency, clearly suggesting that the 100 μH inductor is essentially ideal. However, above 2 MHz, all the factors contributing to R_s start to increase, while the reactance due to the capacitive element C_p is more pronounced. The dropping level of capacitive reactance begins to have a shorting effect across the windings of the inductor and reduces the overall inductive effect. Eventually, if the frequency

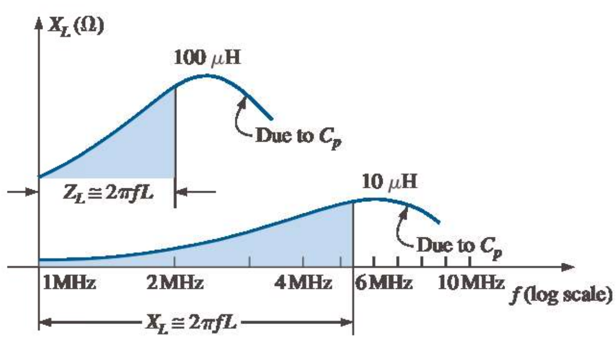


FIG. 14.23
X_L versus frequency for the practical inductor equivalent of Fig. 14.22.

continues to increase, the capacitive effects overcome the inductive effects, and the element actually begins to behave in a capacitive fashion. Note the similarities of this region with the curves in Fig. 14.19. Also, note that decreasing levels of inductance (available with fewer turns and therefore lower levels of C_p) do not demonstrate the degrading effect until higher frequencies are applied.

In general, therefore, the frequency of application for a coil becomes important at increasing frequencies. Inductors lose their ideal characteristics and, in fact, begin to act as capacitive elements with increasing losses at very high frequencies.

Capacitor *C* The capacitor, like the inductor, is not ideal for the full frequency range. In fact, a transition point exists where the characteristics of a capacitor actually take on those of an inductor. The equivalent model for an inductor appearing in Fig. 14.24(a) is an expanded version of that appearing in Fig. 10.21. An inductor L_s was added to reflect the inductance present due to the capacitor leads and any inductance introduced by the design of the capacitor. The inductance of the leads is typically about 0.05 μH per centimeter, which is about 0.2 μH for a capacitor with 2 cm leads at each end—a level of inductance that can be important at very high frequencies.

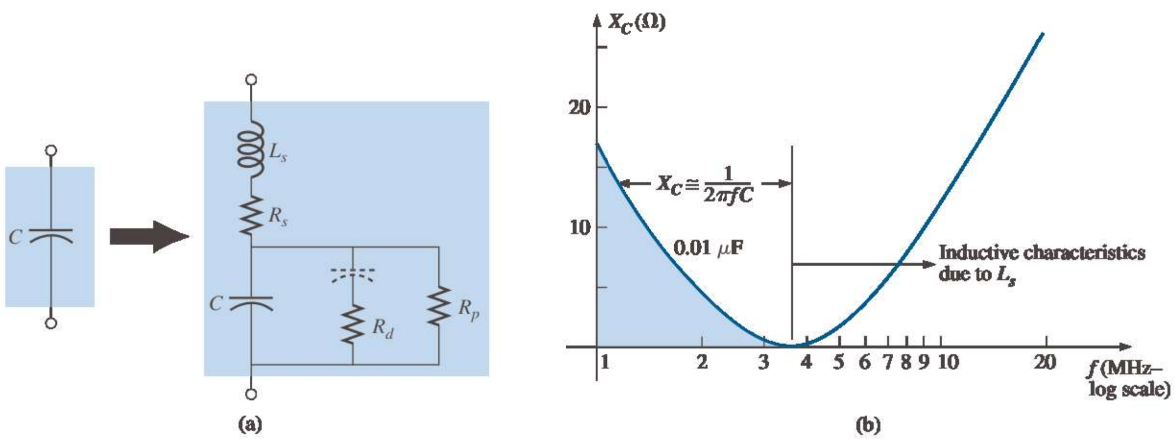


FIG. 14.24
Practical equivalent for a capacitor; (a) network; (b) response.

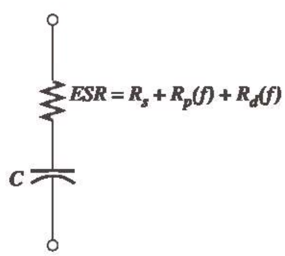
The resistance R_d reflects the energy lost due to molecular friction within the dielectric as the atoms continually realign themselves in the dielectric due to the applied alternating ac voltage. Of interest, however, the relative permittivity decreases with increasing frequencies but eventually undergoes a complete turnaround and begins to increase at very high frequencies. Notice the capacitor included in series with R_d to reflect the fact that this loss is not present under dc conditions. The capacitor assumes its open-circuit state for dc applications.

The resistance R_p, as introduced earlier, is defined by the resistivity of the dielectric (typically 10^{12} Ω or greater) and the case resistance and will determine the level of leakage current if the capacitor is left to discharge. Depending on the capacitor, the discharge time can extend from a few seconds for some electrolytics to hours (paper) or days (polystyrene), revealing that electrolytics typically have much lower levels of R_p than most other capacitors.

The effect of all the elements on the actual response of a 0.01 μF metallized film capacitor with 2 cm leads is provided in Fig. 14.24(b), where the response is almost ideal for the low and mid-frequency range but then at about 3.7 MHz begins to show an inductive response due to L_s.

In general, therefore, the frequency of application is important for capacitive elements because when the frequency increases to a certain level, the element takes on inductive characteristics. Also, the frequency of application defines the type of capacitor (or inductor) that is applied: Electrolytics are limited to frequencies to perhaps 10 kHz, while ceramic or mica can handle frequencies higher than 10 MHz.

The expected temperature range of operation can have an important impact on the type of capacitor chosen for a particular application. Electrolytics, tantalum, and some high-*k* ceramic capacitors are very sensitive to colder temperatures. In fact, most electrolytics lose 20% of their room-temperature capacitance at 0°C (freezing). Higher temperatures (up to 100°C or 212°F) seem to have less impact in general than colder temperatures, but high-*k* ceramics can lose up to 30% of their capacitance level at 100°C compared to room temperature. With experience, you will learn the type of capacitor to use for each application and only be concerned when you encounter very high frequencies, extreme temperatures, or very high currents or voltages.



(a)

(b)

FIG. 14.25
ESR. (a) Impact on equivalent model; (b) Measuring instrument.
[(b) Courtesy of Peak Electronics Design Limited]

ESR The term *equivalent series resistance* (ESR) was introduced in Chapter 10, where it was noted that the topic would surface again after the concept of frequency response was introduced. In the simplest of terms, the ESR as appearing in the simplistic model of Fig. 14.25(a) is the actual dissipative factor one can expect when using a capacitor at various frequencies. For dc conditions it is essentially the dc resistance of the capacitor appearing as R_s in Fig. 14.24(a). However, for any ac application the level of dissipation will be a function of the levels of R_p and R_d and the frequency applied.

Although space does not permit a detailed derivation here, the ESR for a capacitor is defined by the following equation:

$$\text{ESR} = R_s + \frac{1}{\omega^2 C^2 R_p} + \frac{1}{\omega C^2 R_d}$$

Note that the first term is simply the dc resistance and is not a function of frequency. However, the next two terms are a function of frequency in the denominator, revealing that they will increase very quickly as the frequency drops. The result is the valid concern about levels of ESR at low frequencies. At high frequencies, the second two terms will die off quickly, leaving only the dc resistance. In general, therefore, keep in mind that

the level of ESR or equivalent series resistance is frequency sensitive and considerably greater at low frequencies than just the dc resistance. At very high frequencies, it approaches the dc level.

It is such as important factor in some designs that instruments have been developed primarily to measure this quantity. One such instrument appears in Fig. 14.25(b).

There are some general rules about the level of ESR associated with various capacitors. For all applications, the lower the ESR, the better. Electrolytic capacitors typically have much higher levels of ESR than film, ceramic, or paper capacitors. A standard electrolytic 22 μF capacitor may have an ESR between 5 and 30 Ω, while a standard ceramic may have only 10 to 100 mΩ, a significant difference. Electrolytics, however, because of their other characteristics, are still very popular in power supply design—it is simply a matter of balancing the ESR level with other important factors.

EXAMPLE 14.8 At what frequency will the reactance of a 200 mH inductor match the resistance level of a 5 kΩ resistor?

Solution: The resistance remains constant at 5 kΩ for the frequency range of the inductor. Therefore,

$$R = 5000\ \Omega = X_L = 2\pi fL = 2\pi Lf$$
$$= 2\pi(200 \times 10^{-3}\ \text{H})f = 1.257f$$

and
$$f = \frac{5000\ \text{Hz}}{1.257} \cong \mathbf{3.98\ kHz}$$

EXAMPLE 14.9 At what frequency will an inductor of 5 mH have the same reactance as a capacitor of 0.1 μF?

Solution:

$$X_L = X_C$$
$$2\pi fL = \frac{1}{2\pi fC}$$
$$f^2 = \frac{1}{4\pi^2 LC}$$

and

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(5 \times 10^{-3}\ \text{H})(0.1 \times 10^{-6}\ \text{F})}}$$
$$= \frac{1}{2\pi\sqrt{5 \times 10^{-10}}} = \frac{1}{(2\pi)(2.236 \times 10^{-5})} = \frac{10^5\ \text{Hz}}{14.05} \cong \mathbf{7.12\ kHz}$$

14.4 AVERAGE POWER AND POWER FACTOR

A common question is, How can a sinusoidal voltage or current deliver power to a load if it seems to be delivering power during one part of its cycle and taking it back during the negative part of the sinusoidal cycle? The equal oscillations above and below the axis seem to suggest that over one full cycle there is no net transfer of power or energy. However, as mentioned in the last chapter, there is a net transfer of power over one full cycle because power is delivered to the load *at each instant* of the applied voltage or current (except when either is crossing the axis) no matter what the direction is of the current or polarity of the voltage.

To demonstrate this, consider the relatively simple configuration in Fig. 14.26, where an 8 V peak sinusoidal voltage is applied across a 2 Ω resistor. When the voltage is at its positive peak, the power delivered at that instant is 32 W, as shown in the figure. At the midpoint of 4 V, the instantaneous power delivered drops to 8 W; when the voltage crosses the axis, it drops to 0 W. Note, however, that when the applied voltage is at its negative peak, the current may reverse, but at that instant, 32 W is still being delivered to the resistor.

14.8

$L = 200\,mH$

$X_L = R = 5k\Omega$

$X_L \; \Omega$

$X_L = \omega L \Rightarrow 5\times10^3 = 2\pi f \times 200\,m$

$$f = \frac{5\times10^3}{2\pi\times200\times10^{-3}} = 4k\,Hz$$

14, a

$L = 5mH$

$C = 0.1 \mu F$

reactance $X_L = X_C$

$$\omega L = \frac{1}{\omega C}$$

$$\sqrt{\omega^2} = \sqrt{\frac{1}{LC}} \rightarrow \omega = \sqrt{\frac{1}{LC}}$$

$$\frac{2\pi f}{2\pi} = \sqrt{\frac{1}{LC}}$$

$$\Rightarrow f = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{1}{5\times10^{-3} \times 0.1 \times 10^{-6}}} = 7 KHz$$

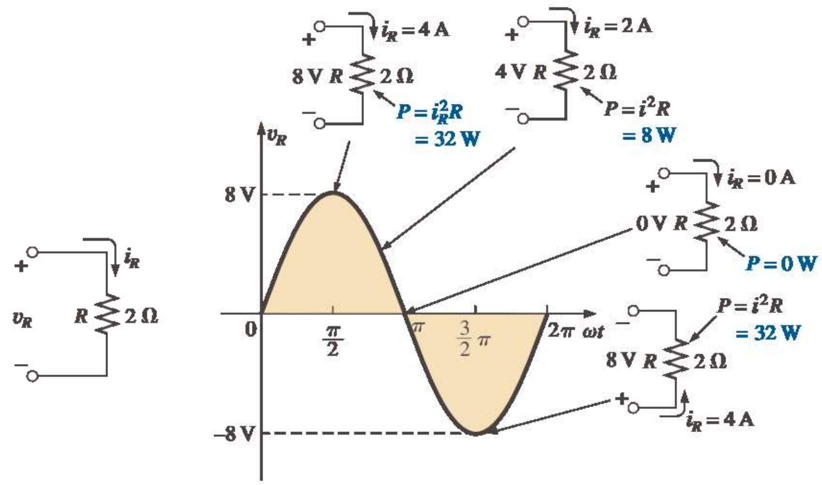


FIG. 14.26
Demonstrating that power is delivered at every instant of a sinusoidal voltage waveform (except $v_R = 0$ V).

In total, therefore,

even though the current through and the voltage across reverse direction and polarity, respectively, power is delivered to the resistive load at each instant of time.

If we plot the power delivered over a full cycle, we obtain the curve in Fig. 14.27. Note that the applied voltage and resulting current are in phase and have twice the frequency of the power curve. For one full cycle of the applied voltage having a period T, the power level peaks for each pulse of the sinusoidal waveform.

The fact that the power curve is always above the horizontal axis reveals that power is being delivered to the load at each instant of time of the applied sinusoidal voltage.

Any portion of the power curve below the axis reveals that power is being returned to the source. The average value of the power curve

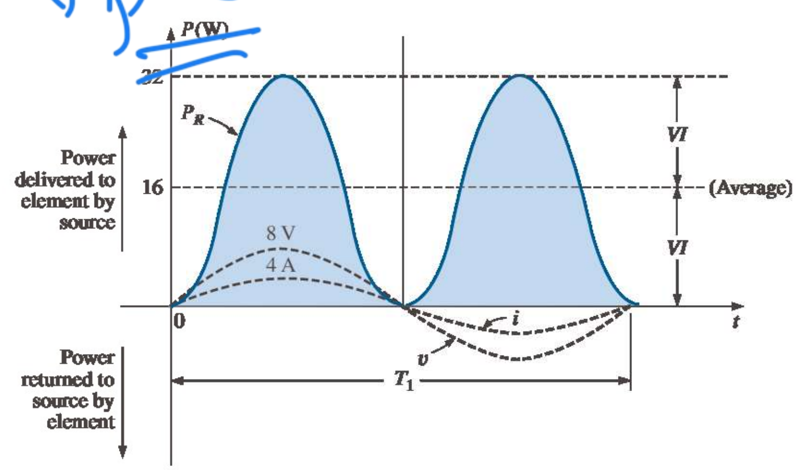


FIG. 14.27
Power versus time for a purely resistive load.

occurs at a level equal to $V_m I_m/2$, as shown in Fig. 14.27. This power level is called the **average** or **real power** level. It establishes a particular level of power transfer for the full cycle, so that we do not have to determine the level of power to apply to a quantity that varies in a sinusoidal nature.

If we substitute the equation for the peak value in terms of the rms value as

$$P_{av} = \frac{V_m I_m}{2} = \frac{(\sqrt{2}\,V_{rms})(\sqrt{2}\,I_{rms})}{2} = \frac{2\,V_{rms} I_{rms}}{2}$$

we find that the average or real power delivered to a resistor takes on the following very convenient form:

$$\boxed{P_{av} = V_{rms} I_{rms}} \qquad \textbf{(14.14)}$$

Note that the power equation is exactly the same when applied to dc networks as long as we work with rms values.

The above analysis was for a purely resistive load. If the sinusoidal voltage is applied to a network with a combination of R, L, and C components, the instantaneous equation for the power levels is more complex. However, if we are careful in developing the general equation and examine the results, we find some general conclusions that will be very helpful in the analysis to follow.

In Fig. 14.28, a voltage with an initial phase angle is applied to a network with any combination of elements that results in a current with the indicated phase angle.

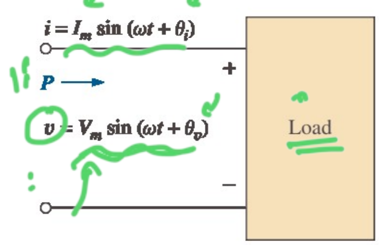


FIG. 14.28
Determining the power delivered in a sinusoidal ac network.

The power delivered at each instant of time is then defined by

$$\begin{aligned} p = vi &= V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i) \\ &= V_m I_m \sin(\omega t + \theta_v)\sin(\omega t + \theta_i) \end{aligned}$$

Using the trigonometric identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

we see that the function $\sin(\omega t + \theta_v)\sin(\omega t + \theta_i)$ becomes

$$\begin{aligned} &\sin(\omega t + \theta_v)\sin(\omega t + \theta_i) \\ &\quad = \frac{\cos[(\omega t + \theta_v) - (\omega t + \theta_i)] - \cos[(\omega t + \theta_v) + (\omega t + \theta_i)]}{2} \\ &\quad = \frac{\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i)}{2} \end{aligned}$$

so that

$$p = \overbrace{\left[\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\right]}^{\text{Fixed value}} - \overbrace{\left[\frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)\right]}^{\text{Time-varying (function of } t)}$$

A plot of v, i, and p on the same set of axes is shown in Fig. 14.29.

Note that the second factor in the preceding equation is a cosine wave with an amplitude of $V_m I_m/2$ and with a frequency twice that of the voltage or current. The average value of this term is zero over one cycle, producing no net transfer of energy in any one direction.

The first term in the preceding equation, however, has a constant magnitude (no time dependence) and therefore provides some net transfer of energy. This term is referred to as the **average power** or **real**

$$P_{ave} = \frac{1}{2} V_m \cdot I_m$$

$$P_{ave} = V_{rms} I_{rms}$$

$$P = V_m I_m \sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$$

$$P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

$$- \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

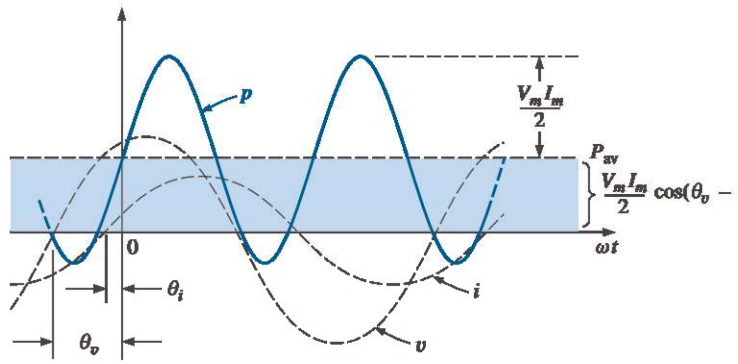


FIG. 14.29
Defining the average power for a sinusoidal ac network.

power as introduced earlier. The angle $(\theta_v - \theta_i)$ is the phase angle between v and i. Since $\cos(-\alpha) = \cos \alpha$,

the magnitude of average power delivered is independent of whether v leads i or i leads v.

Defining θ as equal to $|\theta_v - \theta_i|$, where $|\ |$ indicates that only the magnitude is important and the sign is immaterial, we have

$$P = \frac{V_m I_m}{2}\cos\theta \quad \text{(watts, W)} \tag{14.15}$$

where P is the average power in watts. This equation can also be written

$$P = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

or, since $\quad V_{\text{eff}} = \frac{V_m}{\sqrt{2}} \quad$ and $\quad I_{\text{eff}} = \frac{I_m}{\sqrt{2}}$

Eq. (14.15) becomes

$$P = V_{\text{rms}} I_{\text{rms}} \cos\theta \tag{14.16}$$

Let us now apply Eqs. (14.15) and (14.16) to the basic R, L, and C elements.

Resistor

In a purely resistive circuit, since v and i are in phase, $|\theta_v - \theta_i| = \theta = 0°$, and $\cos\theta = \cos 0° = 1$, so that

$$P = \frac{V_m I_m}{2} = V_{\text{rms}} I_{\text{rms}} \quad \text{(W)} \tag{14.17}$$

or, since $\quad I_{\text{rms}} = \frac{V_{\text{rms}}}{R}$

then

$$P = \frac{V_{\text{rms}}^2}{R} = I_{\text{rms}}^2 R \quad \text{(W)} \tag{14.18}$$

Inductor

In a purely inductive circuit, since v leads i by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos 90° = \frac{V_m I_m}{2}(0) = 0 \text{ W}$$

The average power or power dissipated by the ideal inductor (no associated resistance) is zero watts.

Capacitor

In a purely capacitive circuit, since i leads v by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos(90°) = \frac{V_m I_m}{2}(0) = 0 \text{ W}$$

The average power or power dissipated by the ideal capacitor (no associated resistance) is zero watts.

EXAMPLE 14.10 Find the average power dissipated in a network whose input current and voltage are the following:

$$i = 5 \sin(\omega t + 40°)$$
$$v = 10 \sin(\omega t + 40°)$$

Solution: Since v and i are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

$$P = \frac{V_m I_m}{2} = \frac{(10 \text{ V})(5 \text{ A})}{2} = 25 \text{ W}$$

or $$R = \frac{V_m}{I_m} = \frac{10 \text{ V}}{5 \text{ A}} = 2 \ \Omega$$

and $$P = \frac{V_{\text{rms}}^2}{R} = \frac{[(0.707)(10 \text{ V})]^2}{2} = 25 \text{ W}$$

or $$P = I_{\text{rms}}^2 R = [(0.707)(5 \text{ A})]^2(2) = 25 \text{ W}$$

For the following example, the circuit consists of a combination of resistances and reactances producing phase angles between the input current and voltage different from 0° or 90°.

EXAMPLE 14.11 Determine the average power delivered to networks having the following input voltage and current:

a. $v = 100 \sin(\omega t + 40°)$
   $i = 20 \sin(\omega t + 70°)$
b. $v = 150 \sin(\omega t - 70°)$
   $i = 3 \sin(\omega t - 50°)$

Solutions:

a. $V_m = 100, \quad \theta_v = 40°$
   $I_m = 20 \text{ A}, \quad \theta_i = 70°$
   $\theta = |\theta_v - \theta_i| = |40° - 70°| = |-30°| = 30°$

Ex 14.10

I_m

$i = 5 \sin(\omega t + 40)$ $\quad \theta_i = 40$

$v = 10 \sin(\omega t + 40)$ $\quad \theta_v = 40$

V_m

(R) ✓ $\quad$ L $\quad$ C

$$P = \frac{V_m I_m}{2} = \frac{10 \times 5}{2} = 25\ W$$

$$R = \frac{V_m}{I_m} = \frac{10}{5} = 2\ \Omega$$

$$\theta_v = \theta_i$$

$\Longrightarrow$ Resistor

Ex 14.11

a) $v = 100 \sin(\omega t + 40°)$ → V_m, $\theta_v = 40$

$i = 20 \sin(\omega t + 70°)$ → I_m, $\theta_i = 70$

$$P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

$$= \frac{100 \times 20}{2} \cos(40 - 70)$$

$$P = 866\,W$$

---

b) $v = 150 \sin(\omega t - 70°)$ → V_m, $\theta_v = -70$

$i = 3 \sin(\omega t - 50°)$ → I_m, $\theta_i = -50$

$$P = \frac{150 \times 3}{2} \cos(-70 - (-50))$$

$-70 + 50 = -20$

$\cos(-20)$

$$P = 211.43\,W$$

and

$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(100\text{ V})(20\text{ A})}{2}\cos(30°) = (1000\text{ W})(0.866)$$
$$= \mathbf{866\text{ W}}$$

b. $V_m = 150\text{ V},\quad \theta_v = -70°$
$I_m = 3\text{ A},\quad \theta_i = -50°$
$\theta = |\theta_v - \theta_i| = |-70° - (-50°)|$
$= |-70° + 50°| = |-20°| = 20°$

and

$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(150\text{ V})(3\text{ A})}{2}\cos(20°) = (225\text{ W})(0.9397)$$
$$= \mathbf{211.43\text{ W}}$$

Power Factor

In the equation $P = (V_m I_m/2)\cos\theta$, the factor that has significant control over the delivered power level is the $\cos\theta$. No matter how large the voltage or current, if $\cos\theta = 0$, the power is zero; if $\cos\theta = 1$, the power delivered is a maximum. Since it has such control, the expression was given the name **power factor** and is defined by

$$\text{Power factor} = F_p = \cos\theta \qquad \textbf{(14.19)}$$

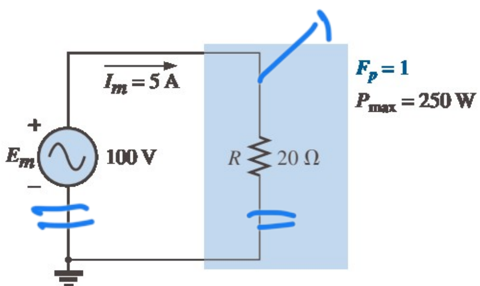


FIG. 14.30
Purely resistive load with $F_p = 1$.

For a purely resistive load such as the one shown in Fig. 14.30, the phase angle between v and i is 0° and $F_p = \cos\theta = \cos 0° = 1$. The power delivered is a maximum of $(V_m I_m/2)\cos\theta = ((100\text{ V})(5\text{ A})/2)(1) = 250\text{ W}$.

For a purely reactive load (inductive or capacitive) such as the one shown in Fig. 14.31, the phase angle between v and i is 90° and $F_p = \cos\theta = \cos 90° = 0$. The power delivered is then the minimum value of zero watts, *even though the current has the same peak value* as that encountered in Fig. 14.30.

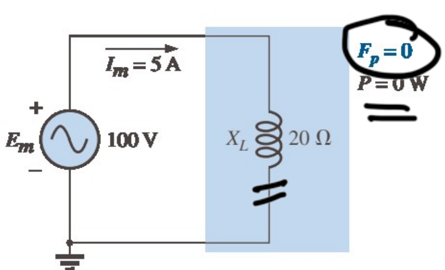


FIG. 14.31
Purely inductive load with $F_p = 0$.

For situations where the load is a combination of resistive and reactive elements, the power factor varies between 0 and 1. The more resistive the total impedance, the closer is the power factor to 1; the more reactive the total impedance, the closer is the power factor to 0.

In terms of the average power and the terminal voltage and current,

$$F_p = \cos\theta = \frac{P}{V_{rms} I_{rms}} \qquad \textbf{(14.20)}$$

The terms *leading* and *lagging* are often written in conjunction with the power factor. *They are defined by the current through the load.* If the current leads the voltage across a load, the load has a **leading power factor.** If the current lags the voltage across the load, the load has a **lagging power factor.** In other words,

capacitive networks have leading power factors, and inductive networks have lagging power factors.

The importance of the power factor to power distribution systems is examined in Chapter 20. In fact, an entire section is devoted to power-factor correction.

$P_{in} = 1000W$ $P_{out} = 1000W$

Power factor = 1 ⌢ 100%,

$P_{in} = 1000W$ $P_{out} = 900W$

Power factor = 0,9

$$\frac{90}{100}$$

Power Factor $Fp = \cos(\theta_v - \theta_i)$

$$Fp = \cos(\theta_v - \theta_i)$$ ✓

$$Fp = \frac{P}{V_{rms} I_{rms}}$$ ✓

leading $\theta_v < \theta_c$

$\Rightarrow$ leading

$\theta_c = 40$

$\theta_v = -20$

lagging $\theta_v > \theta_c$

$\Rightarrow$ lagging

$\theta_v = 80^\circ$

$\theta_c = 30$

EXAMPLE 14.12 Determine the power factors of the following loads, and indicate whether they are leading or lagging:

a. Fig. 14.32
b. Fig. 14.33
c. Fig. 14.34

Solutions:

a. $F_p = \cos\theta = \cos|40° - (-20°)| = \cos 60° = \mathbf{0.5\ leading}$

b. $F_p = \cos\theta\,|80° - 30°| = \cos 50° = \mathbf{0.64\ lagging}$

c. $F_p = \cos\theta = \dfrac{P}{V_{\text{eff}} I_{\text{eff}}} = \dfrac{100\text{ W}}{(20\text{ V})(5\text{ A})} = \dfrac{100\text{ W}}{100\text{ W}} = \mathbf{1}$

The load is resistive, and F_p is neither leading nor lagging.

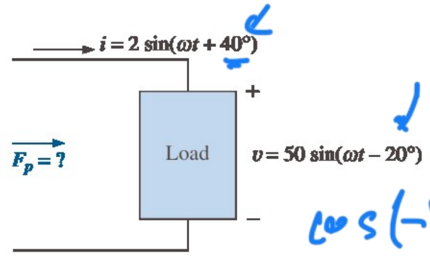


FIG. 14.32
Example 14.12(a).

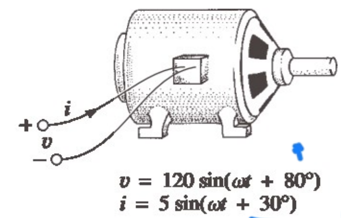


FIG. 14.33
Example 14.12(b).

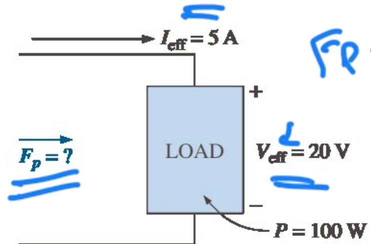


FIG. 14.34
Example 14.12(c).

14.5 COMPLEX NUMBERS

In our analysis of dc networks, we found it necessary to determine the algebraic sum of voltages and currents. Since the same will also be true for ac networks, the question arises, How do we determine the algebraic sum of two or more voltages (or currents) that are continually changing? Although one solution would be to find the algebraic sum on a point-to-point basis (as shown in Section 14.13), this would be a long and tedious process in which accuracy would be directly related to the scale used.

It is the purpose of this chapter to introduce a system of **complex numbers** that, when related to the sinusoidal ac waveform, results in a technique for finding the algebraic sum of sinusoidal waveforms that is quick, direct, and accurate. In the following chapters, the technique is extended to permit the analysis of sinusoidal ac networks in a manner very similar to that applied to dc networks. The methods and theorems as described for dc networks can then be applied to sinusoidal ac networks with little difficulty.

A **complex number** represents a point in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the origin to the point. The horizontal axis is called the *real* axis, while the vertical axis is called the *imaginary* axis. Both are labeled in Fig. 14.35. Every number from zero to $\pm\infty$ can be represented by some point along the real axis. Prior to the development of this system of complex numbers, it was believed that any number not on the real axis did not exist—hence the term *imaginary* for the vertical axis.

In the complex plane, the horizontal or real axis represents all positive numbers to the right of the imaginary axis and all negative numbers to the left of the imaginary axis. All positive imaginary numbers are represented above the real axis, and all negative imaginary numbers, below the real axis. The symbol j (or sometimes i) is used to denote the imaginary component.

Two forms are used to represent a complex number: **rectangular** and **polar.** Each can represent a point in the plane or a radius vector drawn from the origin to that point.

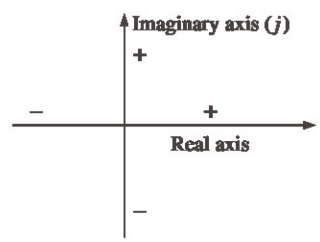


FIG. 14.35
Defining the real and imaginary axes of a complex plane.

14.6 RECTANGULAR FORM

The format for the **rectangular form** is

$$\mathbf{C} = X + jY \qquad \textbf{(14.21)}$$

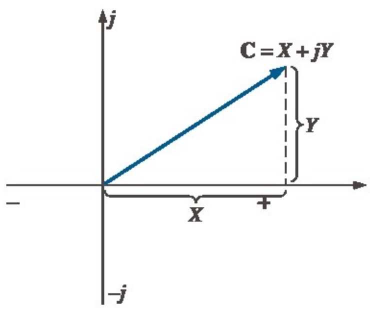


FIG. 14.36
Defining the rectangular form.

as shown in Fig. 14.36. The letter **C** was chosen from the word "complex." The **boldface** notation is for any number with magnitude and direction. The *italic* is for magnitude only.

EXAMPLE 14.13 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 3 + j4$ b. $\mathbf{C} = 0 - j6$ c. $\mathbf{C} = -10 - j20$

Solutions:

a. See Fig. 14.37.
b. See Fig. 14.38.
c. See Fig. 14.39.

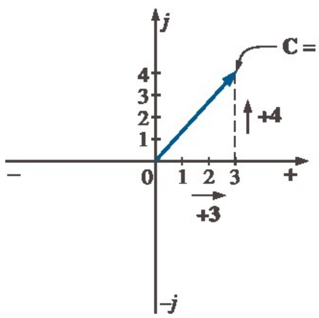


FIG. 14.37
Example 14.13(a).

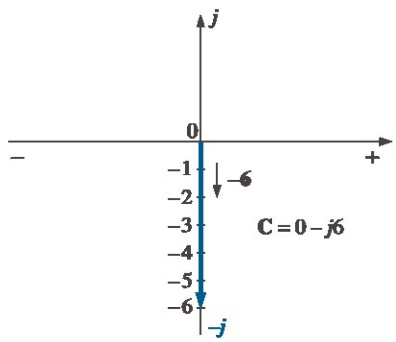


FIG. 14.38
Example 14.13(b).

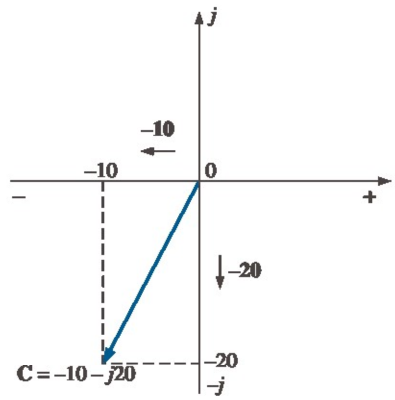


FIG. 14.39
Example 14.13(c).

14.7 POLAR FORM

The format for the **polar form** is

$$\mathbf{C} = Z \angle \theta \tag{14.22}$$

with the letter Z chosen from the sequence X, Y, Z.

Z indicates magnitude only, and θ is *always measured counterclockwise (CCW) from the positive real axis,* as shown in Fig. 14.40. Angles measured in the clockwise direction from the positive real axis must have a negative sign associated with them.

A negative sign in front of the polar form has the effect shown in Fig. 14.41. Note that it results in a complex number directly opposite the

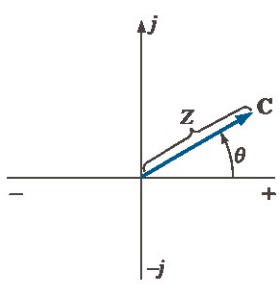


FIG. 14.40
Defining the polar form.

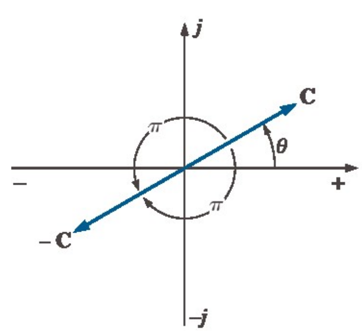


FIG. 14.41
Demonstrating the effect of a negative sign on the polar form.

complex number with a positive sign.

$$-\mathbf{C} = -Z\angle\theta = Z\angle\theta \pm 180° \qquad (14.23)$$

EXAMPLE 14.14 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 5\angle 30°$
b. $\mathbf{C} = 7\angle -120°$
c. $\mathbf{C} = -4.2\angle 60°$

Solutions:

a. See Fig. 14.42.
b. See Fig. 14.43.
c. See Fig. 14.44.

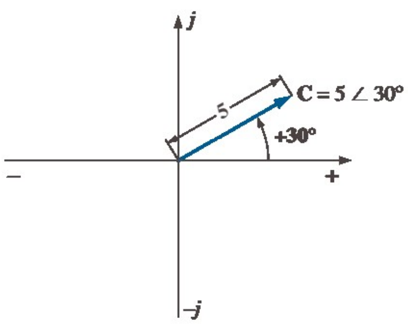


FIG. 14.42
Example 14.14(a).

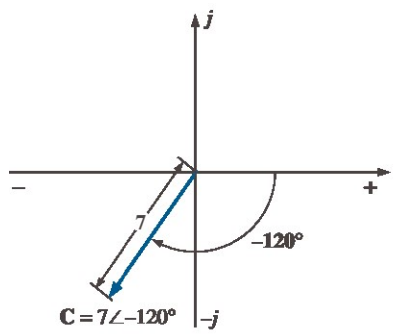


FIG. 14.43
Example 14.14(b).

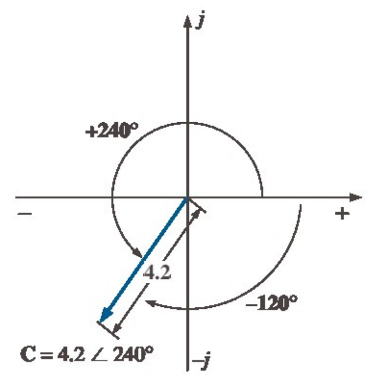


FIG. 14.44
Example 14.14(c).

14.8 CONVERSION BETWEEN FORMS

The two forms are related by the following equations, as illustrated in Fig. 14.45.

Rectangular to Polar

$$Z = \sqrt{X^2 + Y^2} \qquad (14.24)$$

$$\theta = \tan^{-1}\frac{Y}{X} \qquad (14.25)$$

Polar to Rectangular

$$X = Z\cos\theta \qquad (14.26)$$

$$Y = Z\sin\theta \qquad (14.27)$$

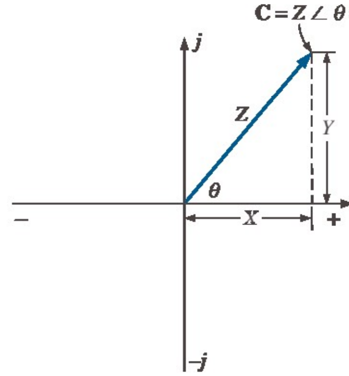


FIG. 14.45
Conversion between forms.

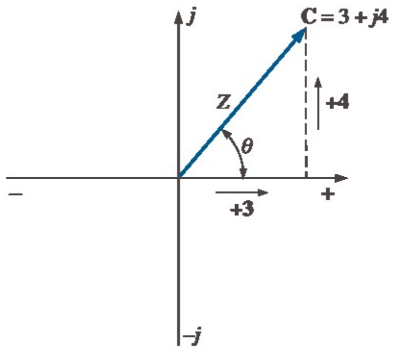


FIG. 14.46
Example 14.15.

EXAMPLE 14.15 Convert the following from rectangular to polar form:

$$\mathbf{C} = 3 + j\,4 \qquad \text{(Fig. 14.46)}$$

Solution:

$$Z = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

and $\quad \mathbf{C = 5\angle 53.13^\circ}$

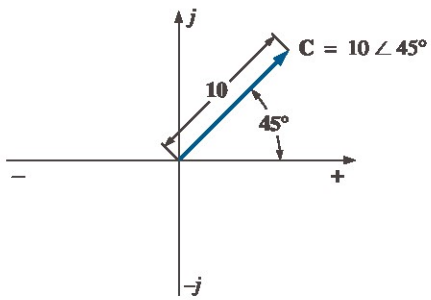


FIG. 14.47
Example 14.16.

EXAMPLE 14.16 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10\,\angle 45^\circ \qquad \text{(Fig. 14.47)}$$

Solution:

$$X = 10\cos 45^\circ = (10)(0.707) = 7.07$$

$$Y = 10\sin 45^\circ = (10)(0.707) = 7.07$$

and $\quad \mathbf{C = 7.97 + j\,7.07}$

If the complex number should appear in the second, third, or fourth quadrant, simply convert it in that quadrant, and carefully determine the proper angle to be associated with the magnitude of the vector.

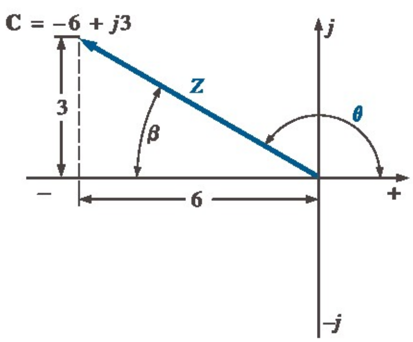


FIG. 14.48
Example 14.17.

EXAMPLE 14.17 Convert the following from rectangular to polar form:

$$\mathbf{C} = -6 + j\,3 \qquad \text{(Fig. 14.48)}$$

Solution:

$$Z = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 6.71$$

$$\beta = \tan^{-1}\left(\frac{3}{6}\right) = 26.57^\circ$$

$$\theta = 180^\circ - 26.57^\circ = 153.43^\circ$$

and $\quad \mathbf{C = 6.71\angle 153.43^\circ}$

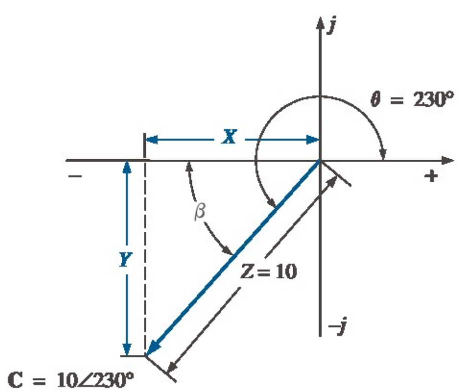


FIG. 14.49
Example 14.18.

EXAMPLE 14.18 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10\,\angle 230^\circ \qquad \text{(Fig. 14.49)}$$

Solution:

$$X = Z\cos\beta = 10\cos(230^\circ - 180^\circ) = 10\cos 50^\circ$$

$$= (10)(0.6428) = 6.428$$

$$Y = Z\sin\beta = 10\sin 50^\circ = (10)(0.7660) = 7.66$$

and $\quad \mathbf{C = -6.43 - j\,7.66}$

14.9 MATHEMATICAL OPERATIONS WITH COMPLEX NUMBERS

Complex numbers lend themselves readily to the basic mathematical operations of addition, subtraction, multiplication, and division. A few basic rules and definitions must be understood before considering these operations.

Let us first examine the symbol j associated with imaginary numbers. By definition,

$$j = \sqrt{-1} \tag{14.28}$$

Thus,

$$j^2 = -1 \tag{14.29}$$

and $\quad j^3 = j^2 j = -1j = -j$

with $\quad j^4 = j^2 j^2 = (-1)(-1) = +1$

$\quad j^5 = j$

and so on. Further,

$$\frac{1}{j} = (1)\left(\frac{1}{j}\right) = \left(\frac{j}{j}\right)\left(\frac{1}{j}\right) = \frac{j}{j^2} = \frac{j}{-1}$$

and

$$\frac{1}{j} = -j \tag{14.30}$$

Complex Conjugate

The **conjugate** or **complex conjugate** of a complex number can be found by simply changing the sign of the imaginary part in the rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of

$$\mathbf{C} = 2 + j\,3$$

is

$$2 - j\,3$$

as shown in Fig. 14.50. The conjugate of

$$\mathbf{C} = 2\,\angle 30°$$

is

$$2\,\angle -30°$$

as shown in Fig. 14.51.

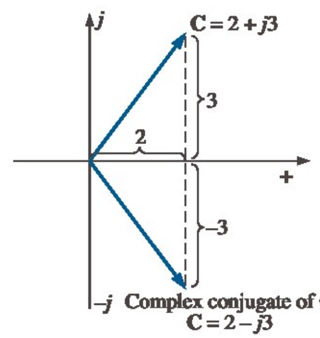


FIG. 14.50
Defining the complex conjugate of a complex number in rectangular form.

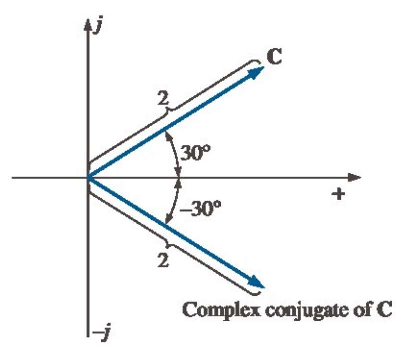


FIG. 14.51
Defining the complex conjugate of a complex number in polar form.

Reciprocal

The **reciprocal** of a complex number is 1 divided by the complex number. For example, the reciprocal of

$$\mathbf{C} = X + jY$$

is

$$\frac{1}{X + jY}$$

and that of $Z\,\angle\theta$ is

$$\frac{1}{Z\,\angle\theta}$$

We are now prepared to consider the four basic operations of *addition, subtraction, multiplication,* and *division* with complex numbers.

Addition

To add two or more complex numbers, add the real and imaginary parts separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\mathbf{C}_1 + \mathbf{C}_2 = (\pm X_1 \pm X_2) + j(\pm Y_1 \pm Y_2) \tag{14.31}$$

There is really no need to memorize the equation. Simply set one above the other and consider the real and imaginary parts separately, as shown in Example 14.19.

EXAMPLE 14.19

a. Add $\mathbf{C}_1 = 2 + j\,4$ and $\mathbf{C}_2 = 3 + j\,1$.
b. Add $\mathbf{C}_1 = 3 + j\,6$ and $\mathbf{C}_2 = -6 + j\,3$.

Solutions:

a. By Eq. (14.31),

$$\mathbf{C}_1 + \mathbf{C}_2 = (2 + 3) + j(4 + 1) = \mathbf{5 + j\,5}$$

Note Fig. 14.52. An alternative method is

$$\begin{array}{c} 2 + j\,4 \\ \underline{3 + j\,1} \\ \downarrow \quad \downarrow \\ \mathbf{5 + j\,5} \end{array}$$

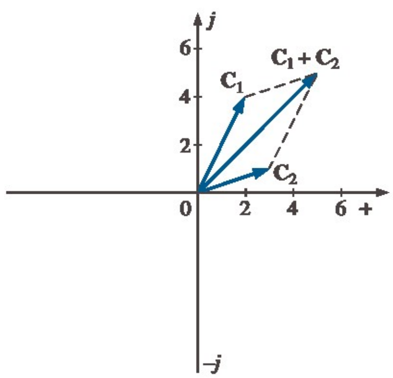


FIG. 14.52
Example 14.19(a).

b. By Eq. (14.31),

$$\mathbf{C}_1 + \mathbf{C}_2 = (3 - 6) + j(6 + 3) = \mathbf{-3 + j\,9}$$

Note Fig. 14.53. An alternative method is

$$\begin{array}{c} 3 + j\,6 \\ \underline{-6 + j\,3} \\ \downarrow \quad \downarrow \\ \mathbf{-3 + j\,9} \end{array}$$

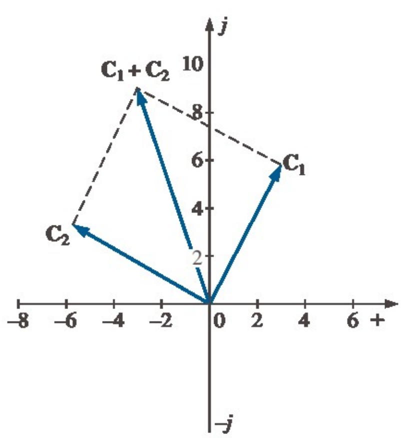


FIG. 14.53
Example 14.19(b).

Subtraction

In subtraction, the real and imaginary parts are again considered separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\mathbf{C}_1 - \mathbf{C}_2 = [\pm X_1 - (\pm X_2)] + j[\pm Y_1 - (\pm Y_2)] \tag{14.32}$$

Again, there is no need to memorize the equation if the alternative method of Example 14.20 is used.

EXAMPLE 14.20

a. Subtract $\mathbf{C}_2 = 1 + j4$ from $\mathbf{C}_1 = 4 + j6$.
b. Subtract $\mathbf{C}_2 = -2 + j5$ from $\mathbf{C}_1 = +3 + j3$.

Solutions:

a. By Eq. (14.32),

$$\mathbf{C}_1 - \mathbf{C}_2 = (4 - 1) + j(6 - 4) = \mathbf{3 + j2}$$

Note Fig. 14.54. An alternative method is

$$\begin{array}{r} 4 + j6 \\ \underline{-(1 + j4)} \\ \downarrow \quad \downarrow \\ \mathbf{3 + j2} \end{array}$$

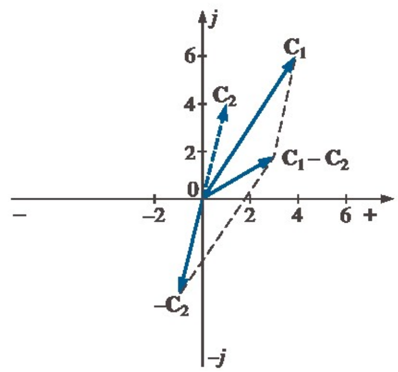


FIG. 14.54
Example 14.20(a).

b. By Eq. (14.32),

$$\mathbf{C}_1 - \mathbf{C}_2 = [3 - (-2)] + j(3 - 5) = \mathbf{5 - j2}$$

Note Fig. 14.55. An alternative method is

$$\begin{array}{r} 3 + j3 \\ \underline{-(-2 + j5)} \\ \downarrow \quad \downarrow \\ \mathbf{5 - j2} \end{array}$$

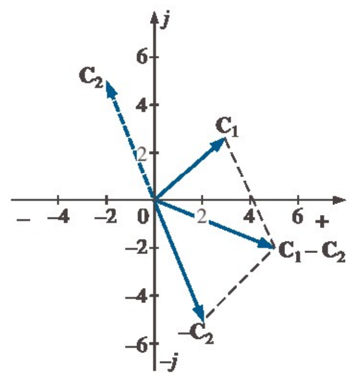


FIG. 14.55
Example 14.20(b).

Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle θ or unless they differ only by multiples of 180°.

EXAMPLE 14.21

a. $2 \angle 45° + 3 \angle 45° = 5 \angle 45°$. Note Fig. 14.56.
b. $2 \angle 0° - 4 \angle 180° = 2 \angle 0° - (-4 \angle 0°) = 6 \angle 0°$. Note Fig. 14.57.

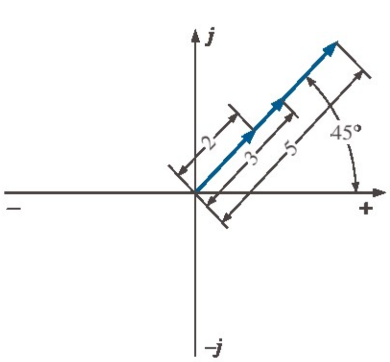


FIG. 14.56
Example 14.21(a).

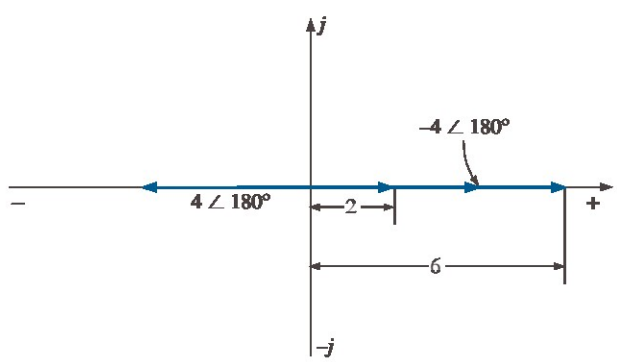


FIG. 14.57
Example 14.21(b).

Multiplication

To multiply two complex numbers in *rectangular* form, multiply the real and imaginary parts of one in turn by the real and imaginary parts of the other. For example, if

$$\mathbf{C}_1 = X_1 + jY_1 \qquad \text{and} \qquad \mathbf{C}_2 = X_2 + jY_2$$

then $\mathbf{C}_1 \cdot \mathbf{C}_2$:

$$\begin{array}{r} X_1 + jY_1 \\ \underline{X_2 + jY_2} \\ X_1X_2 + jY_1X_2 \\ \underline{+ jX_1Y_2 + j^2Y_1Y_2} \\ X_1X_2 + j(Y_1X_2 + X_1Y_2) + Y_1Y_2(-1) \end{array}$$

and

$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = (X_1X_2 - Y_1Y_2) + j(Y_1X_2 + X_1Y_2)} \qquad (14.33)$$

In Example 14.22(b), we obtain a solution without resorting to memorizing Eq. (14.33). Simply carry along the j factor when multiplying each part of one vector with the real and imaginary parts of the other.

EXAMPLE 14.22

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2 + j\,3 \qquad \text{and} \qquad \mathbf{C}_2 = 5 + j\,10$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = -2 - j\,3 \qquad \text{and} \qquad \mathbf{C}_2 = +4 - j\,6$$

Solutions:

a. Using the format above, we have

$$\begin{aligned} \mathbf{C}_1 \cdot \mathbf{C}_2 &= [(2)(5) - (3)(10)] + j[(3)(5) + (2)(10)] \\ &= \mathbf{-20 + j\,35} \end{aligned}$$

b. Without using the format, we obtain

$$\begin{array}{l} -2 - j\,3 \\ \underline{+4 - j\,6} \\ -8 - j\,12 \\ \underline{\quad + j\,12 + j^2\,18} \\ -8 + j(-12 + 12) - 18 \end{array}$$

and $\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{-26 = 26\angle 180^\circ}$

In *polar* form, the magnitudes are multiplied and the angles added algebraically. For example, for

$$\mathbf{C}_1 = Z_1 \angle \theta_1 \qquad \text{and} \qquad \mathbf{C}_2 = Z_2 \angle \theta_2$$

we write

$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = Z_1Z_2 \angle \theta_1 + \theta_2} \qquad (14.34)$$

EXAMPLE 14.23

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 5 \angle 20^\circ \quad \text{and} \quad \mathbf{C}_2 = 10 \angle 30^\circ$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2 \angle -40^\circ \quad \text{and} \quad \mathbf{C}_2 = 7 \angle +120^\circ$$

Solutions:

a. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (5 \angle 20^\circ)(10 \angle 30^\circ) = (5)(10) \angle 20^\circ + 30^\circ = \mathbf{50 \angle 50^\circ}$

b. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (2 \angle -40^\circ)(7 \angle +120^\circ) = (2)(7) \angle -40^\circ + 120^\circ$
$= \mathbf{14 \angle +80^\circ}$

To multiply a complex number in rectangular form by a real number requires that both the real part and the imaginary part be multiplied by the real number. For example,

$$(10)(2 + j\,3) = \mathbf{20 + j\,30}$$

and $\quad 50 \angle 0^\circ(0 + j\,6) = j\,300 = \mathbf{300 \angle 90^\circ}$

Division

To divide two complex numbers in *rectangular* form, multiply the numerator and denominator by the conjugate of the denominator and the resulting real and imaginary parts collected. That is, if

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then
$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(X_1 + jY_1)(X_2 - jY_2)}{(X_2 + jY_2)(X_2 - jY_2)}$$
$$= \frac{(X_1X_2 + Y_1Y_2) + j(X_2Y_1 - X_1Y_2)}{X_2^2 + Y_2^2}$$

and
$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{X_1X_2 + Y_1Y_2}{X_2^2 + Y_2^2} + j\frac{X_2Y_1 - X_1Y_2}{X_2^2 + Y_2^2} \qquad (14.35)$$

The equation does not have to be memorized if the steps above used to obtain it are employed. That is, first multiply the numerator by the complex conjugate of the denominator and separate the real and imaginary terms. Then divide each term by the sum of each term of the denominator squared.

EXAMPLE 14.24

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 1 + j\,4$ and $\mathbf{C}_2 = 4 + j\,5$.
b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = -4 - j\,8$ and $\mathbf{C}_2 = +6 - j\,1$.

Solutions:

a. By Eq. (14.35),

$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(1)(4) + (4)(5)}{4^2 + 5^2} + j\frac{(4)(4) - (1)(5)}{4^2 + 5^2}$$
$$= \frac{24}{41} + \frac{j\,11}{41} \cong \mathbf{0.59 + j\,0.27}$$

b. Using an alternative method, we obtain

$$\begin{array}{l} -4 - j8 \\ \underline{+6 + j1} \\ -24 - j48 \\ \underline{-j4 - j^2 8} \\ -24 - j52 + 8 = -16 - j52 \end{array}$$

$$\begin{array}{l} +6 - j1 \\ \underline{+6 + j1} \\ 36 + j6 \\ \underline{\quad - j6 - j^2 1} \\ 36 + 0 + 1 = 37 \end{array}$$

and
$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{-16}{37} - \frac{j52}{37} = \mathbf{-0.43 - j1.41}$$

To divide a complex number in rectangular form by a real number, both the real part and the imaginary part must be divided by the real number. For example,

$$\frac{8 + j10}{2} = \mathbf{4 + j5}$$

and
$$\frac{6.8 - j0}{2} = 3.4 - j0 = \mathbf{3.4 \angle 0^\circ}$$

In *polar* form, division is accomplished by dividing the magnitude of the numerator by the magnitude of the denominator and subtracting the angle of the denominator from that of the numerator. That is, for

$$\mathbf{C}_1 = Z_1 \angle \theta_1 \qquad \text{and} \qquad \mathbf{C}_2 = Z_2 \angle \theta_2$$

we write
$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{Z_1}{Z_2} \angle \theta_1 - \theta_2} \tag{14.36}$$

EXAMPLE 14.25

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 15 \angle 10^\circ$ and $\mathbf{C}_2 = 2 \angle 7^\circ$.
b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 8 \angle 120^\circ$ and $\mathbf{C}_2 = 16 \angle -50^\circ$.

Solutions:

a. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{15 \angle 10^\circ}{2 \angle 7^\circ} = \frac{15}{2} \angle 10^\circ - 7^\circ = \mathbf{7.5 \angle 3^\circ}$$

b. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{8 \angle 120^\circ}{16 \angle -50^\circ} = \frac{8}{16} \angle 120^\circ - (-50^\circ) = \mathbf{0.5 \angle 170^\circ}$$

We obtain the *reciprocal* in the rectangular form by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{X + jY} = \left(\frac{1}{X + jY}\right)\left(\frac{X - jY}{X - jY}\right) = \frac{X - jY}{X^2 + Y^2}$$

and
$$\frac{1}{X + jY} = \frac{X}{X^2 + Y^2} - j\frac{Y}{X^2 + Y^2} \tag{14.37}$$

In polar form, the reciprocal is

$$\frac{1}{Z\angle\theta} = \frac{1}{Z}\angle-\theta \tag{14.38}$$

A concluding example using the four basic operations follows.

EXAMPLE 14.26 Perform the following operations, leaving the answer in polar or rectangular form:

a. $$\frac{(2 + j3) + (4 + j6)}{(7 + j7) - (3 - j3)} = \frac{(2 + 4) + j(3 + 6)}{(7 - 3) + j(7 + 3)}$$
$$= \frac{(6 + j9)(4 - j10)}{(4 + j10)(4 - j10)}$$
$$= \frac{[(6)(4) + (9)(10)] + j[(4)(9) - (6)(10)]}{4^2 + 10^2}$$
$$= \frac{114 - j24}{116} = \mathbf{0.98 - j0.21}$$

b. $$\frac{(50\angle30^\circ)(5 + j5)}{10\angle-20^\circ} = \frac{(50\angle30^\circ)(7.07\angle45^\circ)}{10\angle-20^\circ} = \frac{353.5\angle75^\circ}{10\angle-20^\circ}$$
$$= 35.35\,\underline{/75^\circ - (-20^\circ)} = \mathbf{35.35\angle95^\circ}$$

c. $$\frac{(2\angle20^\circ)^2(3 + j4)}{8 - j6} = \frac{(2\angle20^\circ)(2\angle20^\circ)(5\angle53.13^\circ)}{10\angle-36.87^\circ}$$
$$= \frac{(4\angle40^\circ)(5\angle53.13^\circ)}{10\angle-36.87^\circ} = \frac{20\angle93.13^\circ}{10\angle-36.87^\circ}$$
$$= 2\,\underline{/93.13^\circ - (-36.87^\circ)} = \mathbf{2.0\angle130^\circ}$$

d. $$3\angle27^\circ - 6\angle-40^\circ = (2.673 + j1.362) - (4.596 - j3.857)$$
$$= (2.673 - 4.596) + j(1.362 + 3.857)$$
$$= \mathbf{-1.92 + j5.22}$$

14.10 CALCULATOR METHODS WITH COMPLEX NUMBERS

The process of converting from one form to another or working through lengthy operations with complex numbers can be time-consuming and often frustrating if one lost minus sign or decimal point invalidates the solution. Fortunately, technologists of today have calculators and computer methods that make the process measurably easier with higher degrees of reliability and accuracy.

Calculators

The TI-89 calculator in Fig. 14.58 is only one of numerous calculators that can convert from one form to another and perform lengthy calculations with complex numbers in a concise, neat form. The basic

FIG. 14.58
TI-89 scientific calculator.
(Don Johnson Photo)

operations with the TI-89 are included primarily to demonstrate the ease with which the conversions can be made and the format for more complex operations.

There are different routes to perform the conversions and operations below, but these instructions give you one approach that is fairly direct and straightforward. Since most operations are in the DEGREE rather than RADIAN mode, the sequence in Fig. 14.59 shows how to set the DEGREE mode for the operations to follow. A similar sequence sets the RADIAN mode if required. The arrows show the direction to scroll. Be aware that it can be a short scroll or a fairly lengthy one. In most cases it is not a single step.

MODE ↓ Angle → ↓ DEGREE ENTER ENTER

FIG. 14.59
Setting the DEGREE mode on the TI-89 calculator.

Rectangular to Polar Conversion The sequence in Fig. 14.60 provides a detailed listing of the steps needed to convert from rectangular to polar form. In the examples to follow, the scrolling steps are not listed to simplify the sequence.

In the sequence in Fig. 14.60, an up scroll is chosen after Matrix because that is a more direct path to Vector ops. A down scroll generates the same result, but it requires going through the whole listing. The sequence seems quite long for such a simple conversion, but with practice you will be able to perform the scrolling steps quite rapidly. Always be sure the input data are entered correctly, such as including the *i* after the *y* component. Any incorrect entry will result in an error listing.

(3 + 5 2ND *i*) 2ND MATH ↓ Matrix →
↑ Vector ops → ↓ ▶ Polar ENTER ENTER **5.83E0 ∠ 59.0E0**

FIG. 14.60
Converting $3 + j5$ to the polar form using the TI-89 calculator.

Polar to Rectangular Conversion The sequence in Fig. 14.61 is a detailed listing of the steps needed to convert from polar to rectangular form. Note in the format that the brackets must surround the polar form. Also, the degree sign must be included with the angle to perform the calculation. The answer is displayed in the engineering notation selected.

(5 2ND ∠ 5 3 . 1 2ND MATH ↓
Angle → ° ENTER) 2ND MATH ↓ Matrix → ↑ Vector ops → ↓
Rect ENTER ENTER **3.00E0+4.00E0i**

FIG. 14.61
Converting $5\angle 53.1°$ to the rectangular form using the TI-89 calculator.

Mathematical Operations Mathematical operations are performed in the natural order of operations, but you must remember to select the format for the solution. For instance, if the sequence in Fig. 14.62 did not include the polar designation, the answer would be in rectangular

(1 0 ∠ 5 0 °) × (2 ∠ 2 0 °)

▶ Polar ENTER **20.00E0 ∠ 70.00E0**

FIG. 14.62
Performing the operation (10 ∠50°)(2 ∠20°).

form even though both quantities in the calculation are in polar form. In the rest of the examples, the scrolling required to obtain mathematical functions is not included to minimize the length of the sequence.

For the product of mixed complex numbers, the sequence of Fig. 14.63 results. Again, the polar form was selected for the solution.

(5 ∠ 5 3 . 1 °) × (2 + 2 *i*)

▶ Polar ENTER ENTER **14.14E0 ∠ 98.10E0**

FIG. 14.63
Performing the operation (5 ∠53.1°)(2 + j 2).

Finally, Example 14.26(c) is entered as shown by the sequence in Fig. 14.64. Note that the results exactly match those obtained earlier.

(2 ∠ 2 0 °) ^ 2 × (3 + 4 *i*)

÷ (8 − 6 *i*) ▶ Polar ENTER ENTER **2.00E0 ∠ 130.0E0**

FIG. 14.64
Verifying the results of Example 14.26(c).

14.11 PHASORS

As noted earlier in this chapter, the addition of sinusoidal voltages and currents is frequently required in the analysis of ac circuits. One lengthy but valid method of performing this operation is to place both sinusoidal waveforms on the same set of axes and add algebraically the magnitudes of each at every point along the abscissa, as shown for $c = a + b$ in Fig. 14.65. This, however, can be a long and tedious process with limited

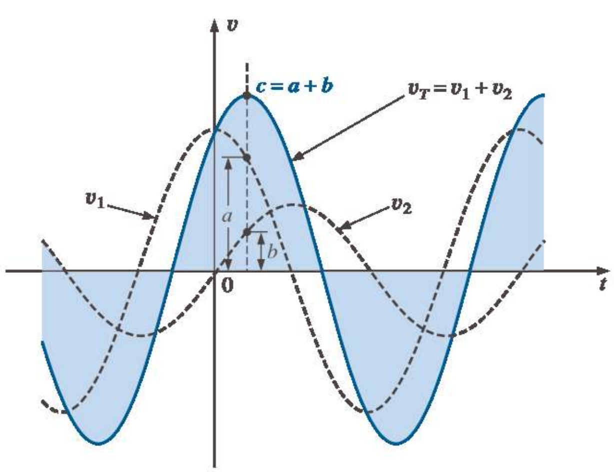


FIG. 14.65
Adding two sinusoidal waveforms on a point-by-point basis.

accuracy. A shorter method uses the rotating radius vector first appearing in Fig. 13.16. This *radius vector,* having a *constant magnitude* (length) with *one end fixed at the origin,* is called a **phasor** when applied to electric circuits.

Because of the importance of the discussion to follow and the benefits it will provide in your future analysis, it is strongly suggested that you return to Section 13.4 and carefully review how the rotating vector of fixed magnitude can generate a sinusoidal waveform at a frequency determined by the speed of rotation of the vector. If the two sinusoidal voltages to be added are in phase, as shown in Fig. 14.66(a), the radius vectors representing each appear on the positive axis at zero degrees because the vertical projection of each at that instant is zero, as shown in Fig. 14.66(b). Note also that the length of each phasor representation is the same as the peak value in Fig. 14.66(a). It should be clear from Fig. 14.66(a) that when the sinusoidal voltages are in phase the sum is simply the sum of the peak values of each as verified in Fig. 14.66(b). In general, therefore,

the addition (or subtraction) of two sinusoidal voltages of the same frequency and phase angle is simply the sum (or difference) of the peak values of each with the sum (or difference) having the same phase angle.

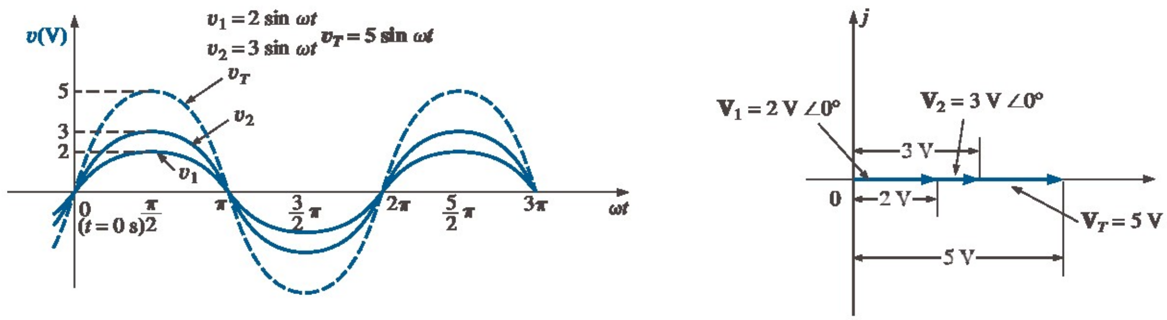
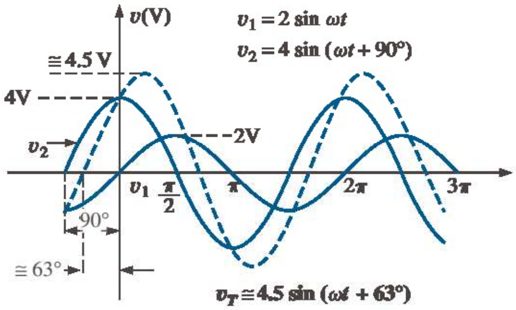


FIG. 14.66
Finding the sum of two sinusoidal waveforms with the same frequency and phase angle.

If the waveforms do not have the same phase angle, a summation of waveforms must be performed as indicated in Fig. 14.65 or using the approach to be described in this section.

Consider the addition of the two sinusoidal voltages of Fig. 14.67(a) out of phase by 90°. The peak value of one is 2 V and the other is 4 V, as shown in Fig. 14.67(a) and in the phasor representation of Fig. 14.67(b). At $t = 0$ s ($\theta = 0°$) the rotating vector of one is passing through the horizontal axis at zero degrees while the other is at its peak value due to the 90° phase shift. If we add the two waveforms of Fig. 14.67(a) on a point-to-point basis, the dashed blue sinusoidal waveform shown in the same figure would result. Note at $\theta = 0°(t = 0$ s) that $v_T = v_1 = 4$ V since $v_1 = 0$ V and at $\theta = \pi/2$ that $v_T = v_1 = 2$ V since $v_2 = 0$ V. The peak value will turn out to be close to 4.1 V at a phase angle of about 76°. It is difficult when adding waveforms to obtain a high level of accuracy unless the graphs are quite large and very carefully drawn. Now, if we look at the **phasor diagram** and simply find the hypotenuse



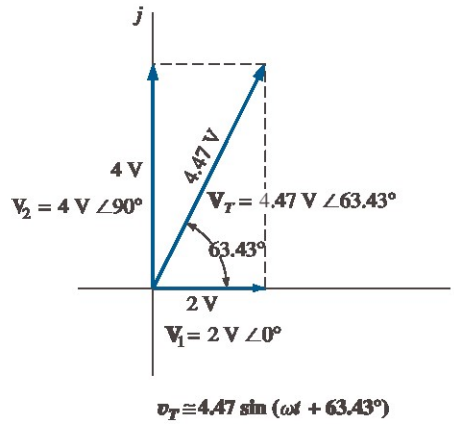


FIG. 14.67
Finding the sum of two sinusoidal waveforms that are out of phase.

of the triangle formed by the two vectors, we find that the magnitude of the projection is also 4.12 V—wonderful. A solution has been found for finding the sum of two sinusoidal waveforms that are not in phase. Simply draw a snapshot of the rotating vectors at $\theta = 0°(t = 0 \text{ s})$ and find the sum of the two vectors. A closer examination of Fig. 14.67(b) also reveals that the phase angle associated with the resultant waveform leads the voltage by 63.43°. In other words, using the phasor diagram we can calculate both the magnitude and phase angle of the sinusoidal waveform representing the sum of the two waveforms. In addition, note the high level of accuracy obtained with a vector addition compared to the artistic approach.

If we now return to Fig. 14.67(b), the phasors representing each sinusoidal waveform can be written as

$$\mathbf{V}_1 = 2 \text{ V} \angle 0° \quad \text{and} \quad \mathbf{V}_2 = 4 \text{ V} \angle 90°$$

Their vector sum then becomes the following using the vector algebra introduced in the previous section. That is,

$$\begin{aligned} \mathbf{V}_T = \mathbf{V}_1 + \mathbf{V}_2 &= 2 \text{ V} \angle 0° + 4 \text{ V} \angle 90° \\ &= 2 \text{ V} + j4 \text{ V} \\ &= 4.47 \text{ V} \angle 63.43° \end{aligned}$$

The result can then be written in the sinusoidal time domain format:

$$v_T = 4.47 \sin(\omega t + 63.43°)$$

If the sinusoidal voltages to be added have different peaks and phase angles, the required calculations are a bit more complex but not extensively so. The next few examples will demonstrate the power of the conclusions just introduced.

EXAMPLE 14.27 Find the sum of the following sinusoidal functions

$$i_1 = 5 \sin(\omega t + 30°)$$
$$i_2 = 6 \sin(\omega t + 60°)$$

a. Using a graphical approach
b. Using a phasor approach

Solutions:

a. The two waveforms and the resultant sum appear in Fig. 14.68. It was obviously a tedious process to add the two waveforms with this approach. Take note that the position of each vector generating the waveforms shown is a snapshot of their position at $\theta = 0°(t = 0\text{ s})$. The sum of the two waveforms is obviously a vector addition of the two waveforms as shown to the left of Fig. 14.68.

b. In phasor form:

$$i_1 = 5\sin(\omega t + 30°) \Rightarrow 5\text{ A}\angle 30°$$
$$i_2 = 6\sin(\omega t + 60°) \Rightarrow 6\text{ A}\angle 60°$$

$$\begin{aligned}\mathbf{I}_T &= \mathbf{I}_1 + \mathbf{I}_2\\ &= 5\text{ A}\angle 30° + 6\text{ A}\angle 60°\\ &= (4.33\text{ A} + j\,2.5\text{ A}) + (3\text{ A} + j\,5.2\text{ A})\\ &= 7.33\text{ A} + j\,7.7\text{ A}\\ &= 10.63\text{ A}\angle 46.41°\end{aligned}$$

and $i_T = \mathbf{10.63\sin(\omega t + 46.41°)}$ as obtained graphically.

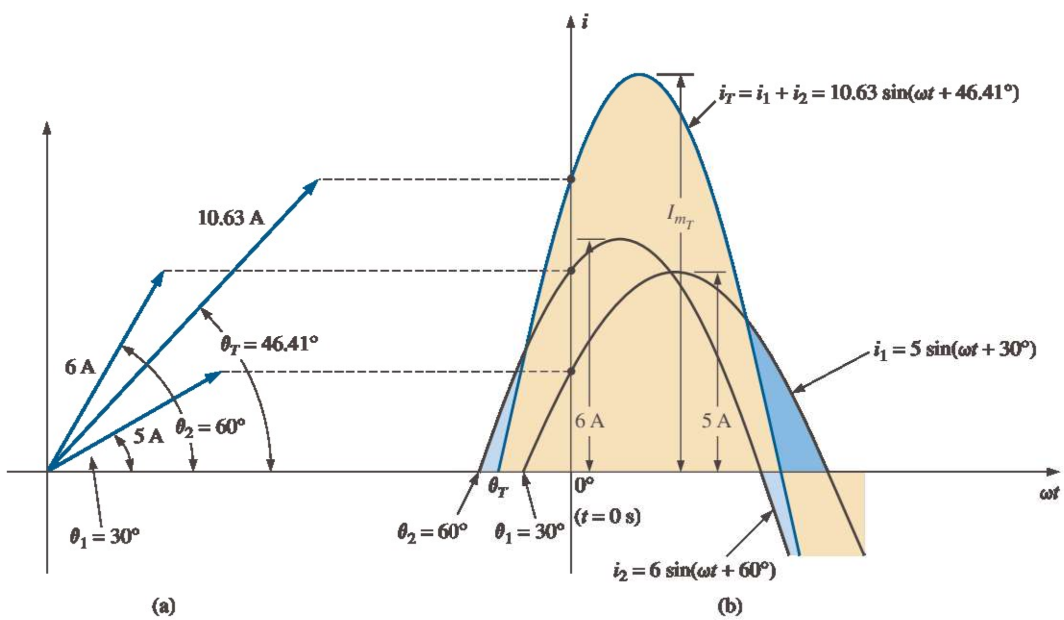


FIG. 14.68
Example 14.27

Since the rms, rather than the peak, values are used almost exclusively in the analysis of ac circuits, the phasor will now be redefined for the purposes of practicality and uniformity as having a magnitude equal to the *rms value* of the sine wave it represents. The angle associated with the phasor will remain as previously described—the phase angle.

In general, for all of the analyses to follow, the phasor form of a sinusoidal voltage or current will be

$$\mathbf{V} = V\angle\theta \qquad \text{and} \qquad \mathbf{I} = I\angle\theta$$

where V and I are rms values and θ is the phase angle. It should be pointed out that in phasor notation, the sine wave is always the reference, and the frequency is not represented.

Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

The use of phasor notation in the analysis of ac networks was first introduced by Charles Proteus Steinmetz in 1897 (Fig. 14.69).

EXAMPLE 14.28 Convert the following from the time to the phasor domain:

Time Domain	Phasor Domain
a. $\sqrt{2}(50)\sin\omega t$	$\mathbf{50\angle 0^\circ}$
b. $69.6\sin(\omega t + 72^\circ)$	$(0.707)(69.6)\angle 72^\circ = \mathbf{49.21\angle 72^\circ}$
c. $45\cos\omega t$	$(0.707)(45)\angle 90^\circ = \mathbf{31.82\angle 90^\circ}$

EXAMPLE 14.29 Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

Phasor Domain	Time Domain
a. $\mathbf{I} = 10\angle 30^\circ$	$i = \sqrt{2}(10)\sin(2\pi 60t + 30^\circ)$ and $\mathbf{i = 14.14\sin(377t + 30^\circ)}$
b. $\mathbf{V} = 115\angle -70^\circ$	$v = \sqrt{2}(115)\sin(377t - 70^\circ)$ and $\mathbf{v = 162.6\sin(377t - 70^\circ)}$

EXAMPLE 14.30 Find the input voltage of the circuit in Fig. 14.70 if

$$\left.\begin{aligned} v_a &= 50\sin(377t + 30^\circ) \\ v_b &= 30\sin(377t + 60^\circ) \end{aligned}\right\} f = 60\text{ Hz}$$

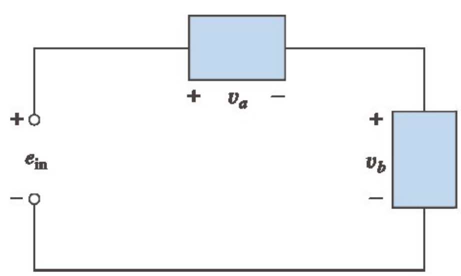


FIG. 14.70
Example 14.30.

Solution: Applying Kirchhoff's voltage law, we have

$$e_{in} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50\sin(377t + 30^\circ) \Rightarrow \mathbf{V}_a = 35.35\text{ V}\angle 30^\circ$$
$$v_b = 30\sin(377t + 60^\circ) \Rightarrow \mathbf{V}_b = 21.21\text{ V}\angle 60^\circ$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35\text{ V}\angle 30^\circ = 30.61\text{ V} + j\,17.68\text{ V}$$
$$\mathbf{V}_b = 21.21\text{ V}\angle 60^\circ = 10.61\text{ V} + j\,18.37\text{ V}$$

FIG. 14.69
Charles Proteus Steinmetz.
Bain News Service/George Grantham Bain Collection/Library of Congress

German-American (Breslau, Germany; Yonkers and Schenectady, NY, USA)
(1865–1923)
Mathematician, Scientist, Engineer, Inventor, Professor of Electrical Engineering and Electrophysics, Union College
Department Head, General Electric Co.

Although the holder of some 200 patents and recognized worldwide for his contributions to the study of hysteresis losses and electrical transients, Charles Proteus Steinmetz is best recognized for his contribution to the study of ac networks. His "Symbolic Method of Alternating-current Calculations" provided an approach to the analysis of ac networks that removed a great deal of the confusion and frustration experienced by engineers of that day as they made the transition from dc to ac systems. His approach (on which the phasor notation of this text is premised) permitted a direct analysis of ac systems using many of the theorems and methods of analysis developed for dc systems. In 1897 he authored the epic work *Theory and Calculation of Alternating Current Phenomena,* which became the authoritative guide for practicing engineers. Dr. Steinmetz was fondly referred to as "The Doctor" at General Electric Company where he worked for some 30 years in a number of important capacities. His recognition as a multigifted genius is supported by the fact that he maintained active friendships with such individuals as Albert Einstein, Guglielmo Marconi, and Thomas A. Edison, to name just a few. He was President of the American Institute of Electrical Engineers (AIEE) and the National Association of Corporation Schools and actively supported his local community (Schenectady) as president of the Board of Education and the Commission on Parks and City Planning.

Then

$$\mathbf{E}_{in} = \mathbf{V}_a + \mathbf{V}_b = (30.61\text{ V} + j\,17.68\text{ V}) + (10.61\text{ V} + j\,18.37\text{ V})$$
$$= 41.22\text{ V} + j\,36.05\text{ V}$$

Converting from rectangular to polar form, we have

$$\mathbf{E}_{in} = 41.22\text{ V} + j\,36.05\text{ V} = 54.76\text{ V}\,\angle 41.17^\circ$$

Converting from the phasor to the time domain, we obtain

$$\mathbf{E}_{in} = 54.76\text{ V}\,\angle 41.17^\circ \Rightarrow e_{in} = \sqrt{2}(54.76)\sin(377t + 41.17^\circ)$$

and $$e_{in} = \mathbf{77.43\,sin(377}t\mathbf{\,+\,41.17^\circ)}$$

A plot of the three waveforms is shown in Fig. 14.71. Note that at each instant of time, the sum of the two waveforms does in fact add up to e_{in}. At $t = 0(\omega t = 0)$, e_{in} is the sum of the two positive values, while at a value of ωt, almost midway between $\pi/2$ and π, the sum of the positive value of v_a and the negative value of v_b results in $e_{in} = 0$.

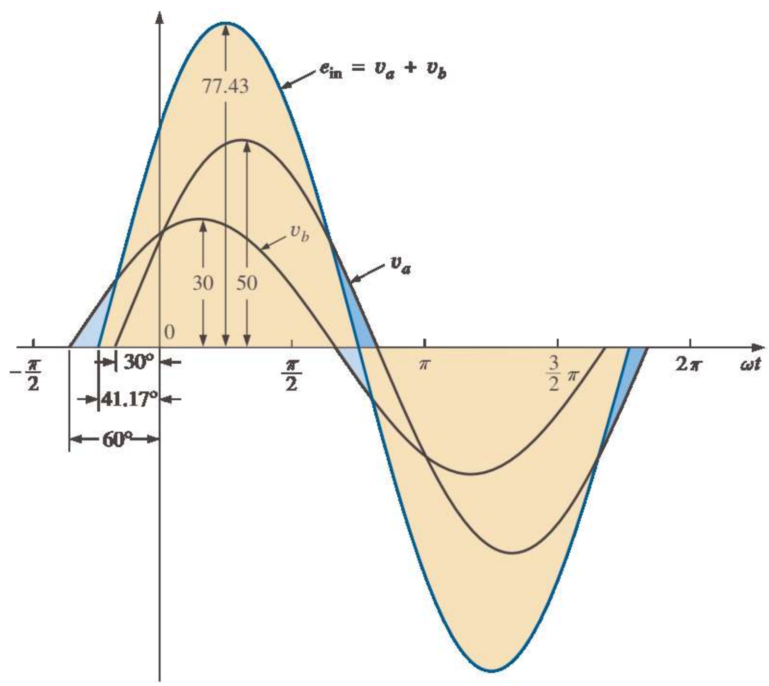


FIG. 14.71
Solution to Example 14.30.

EXAMPLE 14.31 Determine the current i_2 for the network in Fig. 14.72.

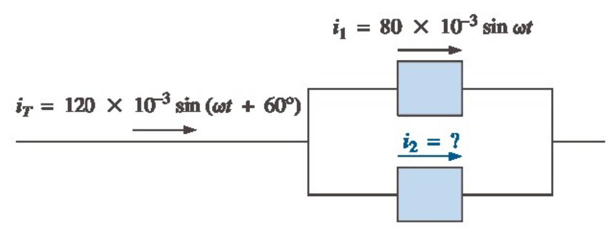


FIG. 14.72
Example 14.31.

Solution: Applying Kirchhoff's current law, we obtain

$$i_T = i_1 + i_2 \qquad \text{or} \qquad i_2 = i_T - i_1$$

Converting from the time to the phasor domain yields

$$i_T = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA } \angle 60°$$
$$i_1 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA } \angle 0°$$

Converting from polar to rectangular form for subtraction yields

$$\mathbf{I}_T = 84.84 \text{ mA } \angle 60° = 42.42 \text{ mA} + j\,73.47 \text{ mA}$$
$$\mathbf{I}_1 = 56.56 \text{ mA } \angle 0° = 56.56 \text{ mA} + j\,0$$

Then

$$\mathbf{I}_2 = \mathbf{I}_T - \mathbf{I}_1$$
$$= (42.42 \text{ mA} + j\,73.47 \text{ mA}) - (56.56 \text{ mA} + j\,0)$$

and $\mathbf{I}_2 = -14.14 \text{ mA} + j\,73.47 \text{ mA}$

Converting from rectangular to polar form, we have

$$\mathbf{I}_2 = 74.82 \text{ mA } \angle 100.89°$$

Converting from the phasor to the time domain, we have

$$\mathbf{I}_2 = 74.82 \text{ mA } \angle 100.89° \Rightarrow$$
$$i_2 = \sqrt{2}(74.82 \times 10^{-3})\sin(\omega t + 100.89°)$$

and $\mathbf{i_2 = 105.8 \times 10^{-3} \sin(\omega t + 100.89°)}$

A plot of the three waveforms appears in Fig. 14.73. The waveforms clearly indicate that $i_T = i_1 + i_2$.

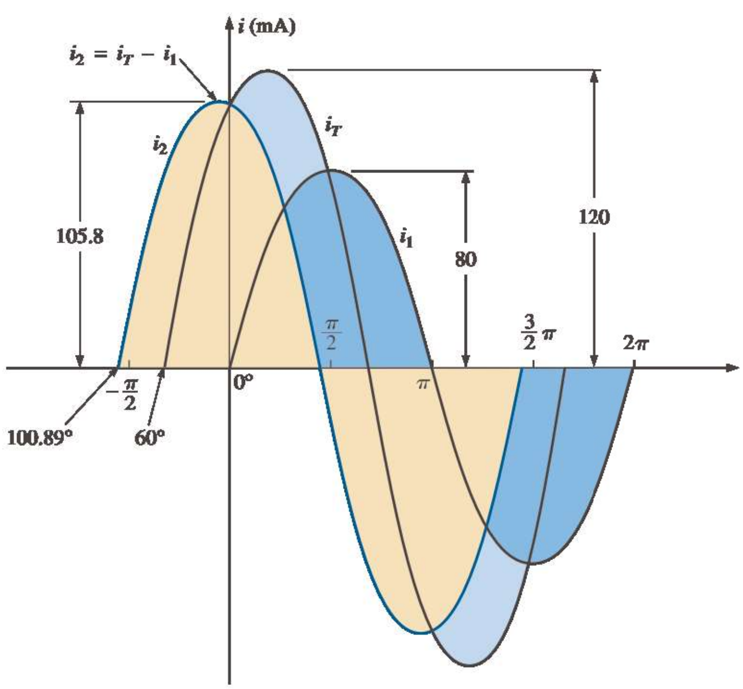


FIG. 14.73
Solution to Example 14.31.

14.12 COMPUTER ANALYSIS

PSpice

Capacitors and the ac Response The simplest of ac capacitive circuits is now analyzed to introduce the process of setting up an ac source and running an ac transient simulation. The ac source in Fig. 14.74 is obtained through **Place part** key-**SOURCE-VSIN-OK.** Change the name or value of any parameter by double-clicking on the parameter on the display. The peak value **(VAMPL)** of the source voltage is 5 V, the frequency 1 kHz, and the phase angle zero degrees.

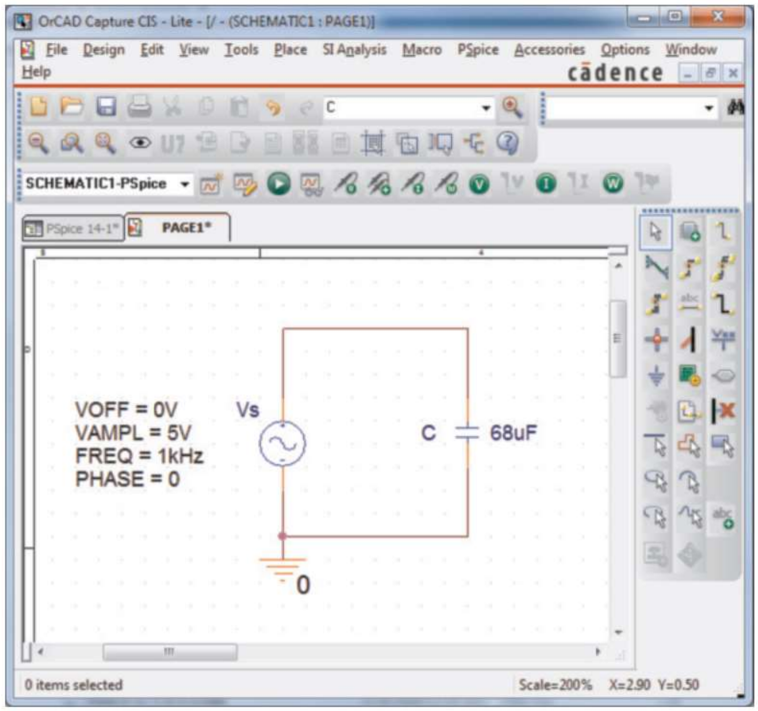


FIG. 14.74

Using PSpice to analyze the response of a capacitor to a sinusoidal ac signal.

The simulation process is initiated by selecting the **New Simulation Profile.** Under **New Simulation,** enter **PSpice 14-1** for the **Name** followed by **Create.** The result will be a blinking **Simulation Setting-PSpice 14-1** dialog box at the bottom of the window that can be deposited on the screen by simply clicking on the dialog box. In the **Simulation Settings** dialog box, select **Analysis** and choose **Time Domain(Transient)** under **Analysis type.** Set the **Run to time** at 3 ms to permit a display of three cycles of the sinusoidal waveforms ($T = 1/f = 1/1000\text{ Hz} = 1\text{ ms}$). Leave the **Start saving data after** at 0 s, and set the **Maximum step size** at $3\text{ ms}/1000 = 3\ \mu s$. Clicking **OK** and then selecting the **Run PSpice** icon results in a **SCHEMATIC1-PSpice 14-1** dialog box at the bottom of the window that can be deposited on the screen by simply clicking on the dialog box. The resulting plot has a horizontal axis that extends from 0 to 3ms.

Now you must tell the computer which waveforms you are interested in. First, take a look at the applied ac source by selecting **Trace-Add Trace-V(Vs:+)** followed by **OK.** The result is the sweeping ac voltage in the bottom region of the screen in Fig. 14.75. Note that it has a peak

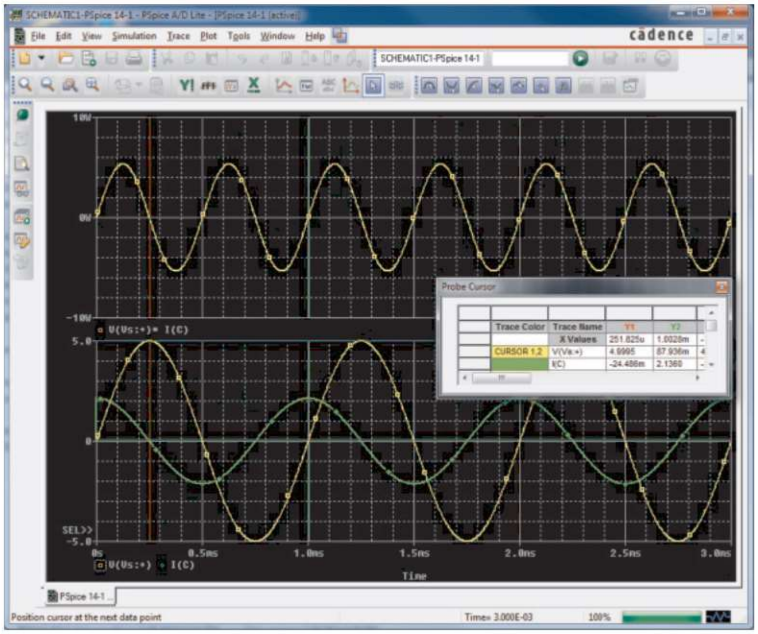


FIG. 14.75

A plot of the voltage, current, and power for the capacitor in Fig. 14.74.

value of 5 V, and three cycles appear in the 3 ms time frame. The current for the capacitor can be added by selecting **Trace-Add Trace** and choosing **I(C)** followed by **OK.** The resulting waveform for I(C) appears at a 90° phase shift from the applied voltage, with the current leading the voltage (the current has already peaked as the voltage crosses the 0 V axis). Since the peak value of each plot is in the same magnitude range, the 5 appearing on the vertical scale can be used for both. A theoretical analysis results in $X_C = 2.34\ \Omega$, and the peak value of $I_C = E/X_C = 5\text{ V}/2.34 = 2.136\text{ A}$, as shown in Fig. 14.75.

For practice, let us obtain the curve for the power delivered to the capacitor over the same time period. First select **Plot-Add Plot to Window-Trace-Add Trace** to obtain the **Add Traces** dialog box. Select **W(C)** followed by **OK** and the top plot of Fig. 14.75 will appear showing that over time the net power delivered is zero (the average value). The power to the capacitor can also be found by first choosing **V(Vs: +)** followed by ***** from the **Function** listing on the right side of the **Add Traces** dialog box and then **I(C).** The result is the expression **V(Vs: +)*I(C)** of the power format: $p = vi$. Click **OK,** and the power plot at the top of Fig. 14.75 appears. Note that over the full three cycles, the area above the axis equals the area below—there is no net transfer of power over the 3 ms period. Note also that the power curve is sinusoidal (which is quite interesting) with a frequency twice that of the applied signal. Using the cursor control, we can determine that the maximum power (peak value of the sinusoidal waveform) is 5.34 W. The cursors, in fact, have been added to the lower curves to show the peak value of the applied sinusoid and the resulting current.

After selecting the **Toggle cursor** icon, left-click to surround the symbol to the left of **V(Vs: +)** at the bottom of the plot with a dashed line to establish that the cursor is providing the levels of that quantity. Then a left-click on the plot will establish the cursor option. When placed at $\frac{1}{4}$ of the total period (250 μs), the peak value is approximately

5 V (Y_1) as shown in the **Probe Cursor** dialog box. Placing the cursor over the symbol next to **I(C)** at the bottom of the plot and right-clicking assigns the right cursor to the current. Placing it at exactly 1 ms (**Y2**) results in a peak value of 2.136 A to match the solution above. To further distinguish between the voltage and current waveforms, the color and the width of the lines of the traces were changed. With the **Toggle cursor** key disabled, place the cursor right on the plot line and right-click. The **Properties** option appears. When **Properties** is selected, a **Trace Properties** dialog box appears in which the yellow color can be selected and the width widened to improve the visibility on the black background. Note that yellow was chosen for **Vs** and green for **I(C).** Note also that the axis and the grid have been changed to a more visible color using the same procedure.

Multisim

Since PSpice reviewed the response of a capacitive element to an ac voltage, Multisim repeats the analysis for an inductive element. The ac voltage source was derived from the **Place Source** parts bin as described in Chapter 13 with the values appearing in Fig. 14.76 set in the **AC-Voltage** dialog box.

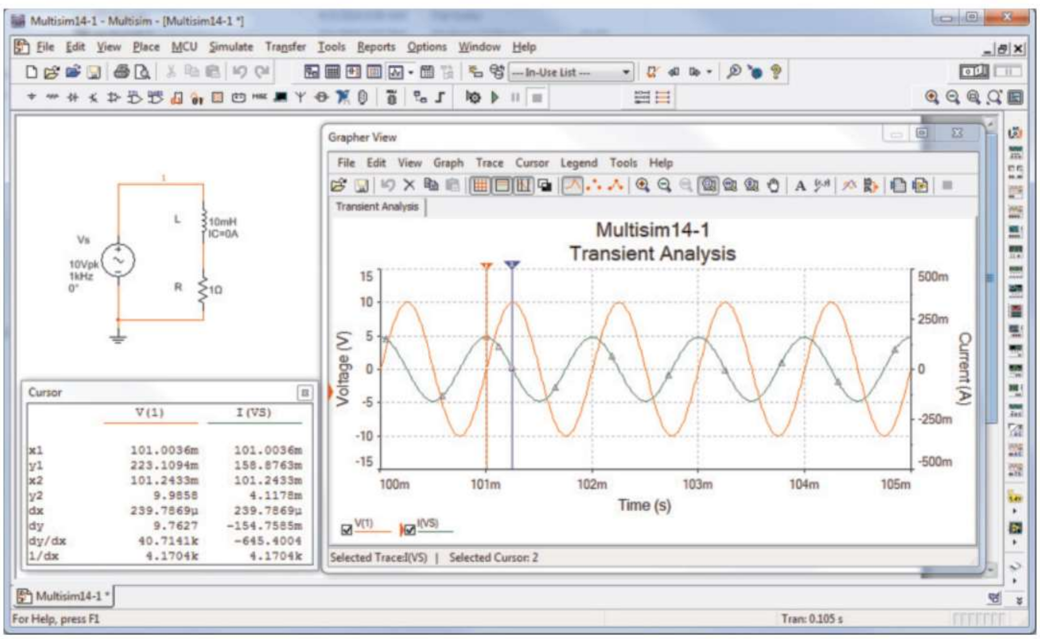


FIG. 14.76
Using Multisim to review the response of an inductive element to a sinusoidal ac signal.

Once the circuit has been constructed, the sequence **Simulate-Analyses-Transient Analysis** results in a **Transient Analysis** dialog box. Select **Analysis** parameters and set **Start Time** to 0 s and **End Time** to 105 ms using 0.105 s or 105E-3 s. Then select **Analysis options** and set maximum number of points to 10,000 to ensure a good display for the rapidly changing waveform. The 105 ms was set as the **End Time** to give the network 100 ms to settle down in its steady-state mode and 5 ms for five cycles in the output display.

Next the **Output** heading was chosen within the dialog box, and the source voltage **V(1)** and source current **I(VS)** were moved from the **Variables in Circuit** to **Selected variables for analysis** using the **Add** option. Choosing **Simulate** results in a waveform that extends from 0 s to 105 ms. Even though we plan to save only the response that occurs after 100 ms, the computer is unaware of our interest, and it plots the response for the entire period. This is corrected by selecting **Trace-Trace Properties** to obtain the **Graph Properties** dialog box. Selecting **Bottom Axis** permits setting the **Range** from a **Minimum of 0.100 s = 100 ms** to a **Maximum of 0.105 s = 105 ms.** Click **OK,** and the time period of Fig. 14.76 is displayed. The grid structure is added by selecting the **Show Grid** keypad, and the color associated with each curve is displayed if we choose **Legend-Show Legend.**

It is clear from the plot that the scale for the source current has to be improved for us to be able to clearly read its peak and negative values. This is done by first clicking on the **I(VS)** curve to set the **Selected Trace** at the bottom of the graph as **I(VS).** A right click, and one can choose the **Properties** option to obtain the **Graph Properties** dialog box. Under **Traces,** select **Right axis** under **Y-vertical axis.** Then select **Right Axis** to establish the right axis as the scale to be used for the source current. Insert the **Label: Current(A),** select **Enabled** under the **Axis** heading, and finally choose **Pen Size** as 1. The **Scale** is **Linear** and of range -0.5 to 0.5 (-500 mA to 500 mA), with **Total Ticks** of 8 and **Minor Ticks** of 2. The result is the plot of Fig. 14.76. The right axis can now be improved by selecting **Graph Properties** again, followed by **Left Axis,** whereby the **Current(A)** can be deleted. We can now see that the source current has a peak value of about 160 mA. For more detail on the waveforms, select **Cursor-Show Cursors** to obtain the **Transient Analysis** dialog box with box **V(1)** and **I(VS)** listed with the same color headings as used on the graph. Clicking on one of the cursors and moving it horizontally to the maximum value of the current will result in **x1** = 101.0 ms with **y1** at 158.88 mA. Actually, the **max y** appears below at 159.07 mA, which could have been obtained if we had increased the number of data points. Moving the other cursor to find the minimum value of current will result in **x2** = 101.24 ms with **y2** at 4.1 mA (the closest to the level of 0 mA obtainable with this data level setting). The maximum value of **V(1)** appears below as 9.986 V $\cong$ 10 V (at **x1** = 101 ms), which it should be, and the distance between the maximum value of **I(VS)** and the its minimum value is **dx** = 239.79 μs, which is very close to 0.25 ms, or one fourth of the period of the applied signal.

PROBLEMS

SECTION 14.1 Introduction

1. Plot the following waveform versus time showing one clear, complete cycle. Then determine the derivative of the waveform using Eq. (14.1), and sketch one complete cycle of the derivative directly under the original waveform. Compare the magnitude of the derivative at various points versus the slope of the original sinusoidal function.

$$v = 4 \sin 62.8t$$

2. Repeat Problem 1 for the following sinusoidal function, and compare results. In particular, determine the frequency of the waveforms of Problems 1 and 2, and compare the magnitude of the derivative.

$$v = 10 \sin 377t$$

3. What is the derivative of each of the following sinusoidal expressions?
   a. $10 \sin 377t$
   b. $20 \sin(400t + 60°)$
   c. $\sqrt{2}\,20 \sin(157t - 20°)$
   d. $-200 \sin(t + 180°)$

SECTION 14.2 Response of Basic *R*, *L*, and *C* Elements to a Sinusoidal Voltage or Current

4. The voltage across a 20 Ω resistor is as indicated. Find the sinusoidal expression for the current. In addition, sketch the v and i sinusoidal waveforms on the same axis.
   a. $160 \sin 100t$
   b. $60 \sin(2000t + 45°)$
   c. $6 \cos(\omega t + 10°)$
   d. $-12 \sin(\omega t + 40°)$
5. The current through a 7.8 kΩ resistor is as indicated. Find the sinusoidal expression for the voltage. In addition, sketch the v and i sinusoidal waveforms on the same axis.
   a. $0.2 \sin 500t$
   b. $5 \times 10^{-3} \sin(600t - 120°)$
6. Determine the inductive reactance (in ohms) of a 3 mH coil for
   a. dc

   and for the following frequencies:
   b. 60 Hz
   c. 8 kHz
   d. 1.4 MHz
7. Determine the closest standard value inductance that has a reactance of
   a. 2.5 kΩ at $f = 12.47$ kHz.
   b. 45 kΩ at $f = 5.8$ kHz.
8. Determine the frequency at which a 47 mH inductance has the following inductive reactances:
   a. 10 Ω
   b. 4 kΩ
   c. 12 kΩ
9. The current through a 20 Ω inductive reactance is given. What is the sinusoidal expression for the voltage? Sketch the v and i sinusoidal waveforms on the same axis.
   a. $i = 25 \times 10^{-3} \sin 200t$
   b. $i = 40 \times 10^{-3} \sin(\omega t + 60°)$
   c. $i = -6 \sin(\omega t - 30°)$
10. The current through a 0.15 H coil is given. What is the sinusoidal expression for the voltage?
    a. $15 \sin 150t$
    b. $6 \times 10^{-6} \sin(400t + 20°)$
11. The voltage across a 40 Ω inductive reactance is given. What is the sinusoidal expression for the current? Sketch the v and i sinusoidal waveforms on the same set of axes.
    a. $120 \sin \omega t$
    b. $30 \sin(\omega t + 20°)$
12. The voltage across a 0.25 H coil is given. What is the sinusoidal expression for the current?
    a. $2.5 \sin 90t$
    b. $16 \times 10^{-3} \sin(20t + 5°)$
13. Determine the capacitive reactance (in ohms) of a 0.4 μF capacitor for
    a. dc

    and for the following frequencies:
    b. 80 Hz
    c. 2.5 kHz
    d. 2.5 MHz
14. Determine the closest standard value capacitance that has a reactance of
    a. 75 Ω at $f = 250$ Hz.
    b. 2.2 kΩ at 36 kHz.
15. Determine the frequency at which a 3.9 μF capacitor has the following capacitive reactances:
    a. 10 Ω b. 60 kΩ
    c. 0.1 Ω d. 2000 Ω
16. The voltage across a 2.5 Ω capacitive reactance is given. What is the sinusoidal expression for the current? Sketch the v and i sinusoidal waveforms on the same set of axes.
    a. $120 \sin \omega t$
    b. $4 \times 10^{-3} \sin(\omega t + 40°)$
17. The voltage across a 1 μF capacitor is given. What is the sinusoidal expression for the current?
    a. $30 \sin 250t$
    b. $90 \times 10^{-3} \sin 377t$
18. The current through a 2 kΩ capacitive reactance is given. Write the sinusoidal expression for the voltage. Sketch the v and i sinusoidal waveforms on the same set of axes.
    a. $i = 50 \times 10^{-3} \sin \omega t$
    b. $i = 2 \times 10^{-6} \sin(\omega t + 60°)$
19. The current through a 0.50 μF capacitor is given. What is the sinusoidal expression for the voltage?
    a. $0.20 \sin 500t$
    b. $5 \times 10^{-3} \sin(377t - 45°)$

*20. For the following pairs of voltages and currents, indicate whether the element involved is a capacitor, an inductor, or a resistor, and find the value of *C*, *L*, or *R* if sufficient data are given:
   a. $v = 550 \sin(377t + 50°)$
      $i = 11 \sin(377t - 40°)$
   b. $v = 36 \sin(754t - 80°)$
      $i = 4 \sin(754t - 170°)$
   c. $v = 10.5 \sin(\omega t - 13°)$
      $i = 1.5 \sin(\omega t - 13°)$

*21. Repeat Problem 20 for the following pairs of voltages and currents with $\omega = 157$ rad/s.
   a. $v = 2000 \sin \omega t$
      $i = 5 \cos \omega t$
   b. $v = 80 \sin(157t + 150°)$
      $i = 2 \sin(157t + 60°)$
   c. $v = 35 \sin(\omega t - 20°)$
      $i = 7 \cos(\omega t - 110°)$

SECTION 14.3 Frequency Response of the Basic Elements

22. Plot X_L versus frequency for a 3 mH coil using a frequency range of zero to 100 kHz on a linear scale.
23. Plot X_C versus frequency for a 1 μF capacitor using a frequency range of zero to 10 kHz on a linear scale.
24. At what frequency will the reactance of a 1.5 μF capacitor equal the resistance of a 2 kΩ resistor?
25. The reactance of a coil equals the resistance of a 10 kΩ resistor at a frequency of 5 kHz. Determine the inductance of the coil.

26. Determine the frequency at which a 2 μF capacitor and an 80 mH inductor will have the same reactance.

27. Determine the capacitance required to establish a capacitive reactance that will match that of a 2 mH coil at a frequency of 60 kHz.

SECTION 14.4 Average Power and Power Factor

*28. Find the average power loss and power factor for each of the circuits whose input current and voltage are as follows:
   a. $v = 60 \sin(\omega t + 30°)$
      $i = 15 \sin(\omega t + 60°)$
   b. $v = -50 \sin(\omega t - 20°)$
      $i = -2 \sin(\omega t - 20°)$
   c. $v = 50 \sin(\omega t + 80°)$
      $i = 3 \cos(\omega t - 20°)$
   d. $v = 75 \sin(\omega t - 5°)$
      $i = 0.08 \sin(\omega t + 35°)$

29. If the current through and voltage across an element are $i = 8 \sin(\omega t + 40°)$ and $v = 56 \sin(\omega t + 50°)$, respectively, compute the power by I^2R, $(V_m I_m/2)\cos\theta$, and $VI\cos\theta$, and compare answers.

30. A circuit dissipates 150 W (average power) at 200 V (effective input voltage) and 2.5 A (effective input current). What is the power factor? Repeat if the power is 0 W; 500 W.

*31. The power factor of a circuit is 0.5 lagging. The power delivered in watts is 600. If the input voltage is $60 \sin(\omega t + 20°)$, find the sinusoidal expression for the input current.

32. In Fig. 14.77, $e = 120 \sin(2\pi 60t + 20°)$.
   a. What is the sinusoidal expression for the current?
   b. Find the power loss in the circuit.
   c. How long (in seconds) does it take the current to complete six cycles?

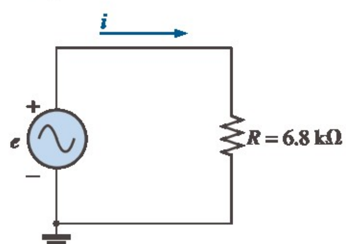


FIG. 14.77
Problem 32.

33. In Fig. 14.78, $e = 240 \sin(1500t + 45°)$.
   a. Find the sinusoidal expression for i.
   b. Find the average power loss by the inductor.

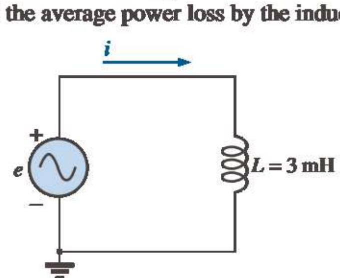


FIG. 14.78
Problem 33.

34. In Fig. 14.79, $i = 20 \times 10^{-3} \sin(2\pi 600t - 30°)$.
   a. Find the sinusoidal expression for e.
   b. Find the average power loss in the capacitor.

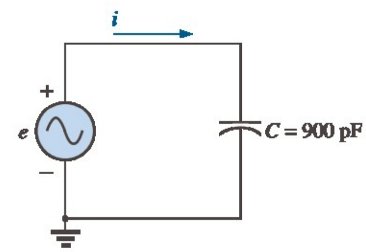


FIG. 14.79
Problem 34.

*35. For the network in Fig. 14.80 and the applied signal:
   a. Determine the sinusoidal expressions for i_1 and i_2.
   b. Find the sinusoidal expression for i_s by combining the two parallel capacitors.

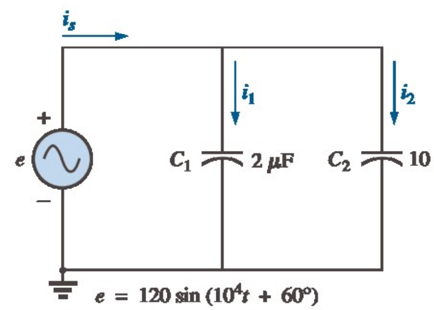


FIG. 14.80
Problem 35.

*36. For the network in Fig. 14.81 and the applied source:
   a. Determine the sinusoidal expression for the source voltage v_s.
   b. Find the sinusoidal expression for the currents i_1 and i_2.

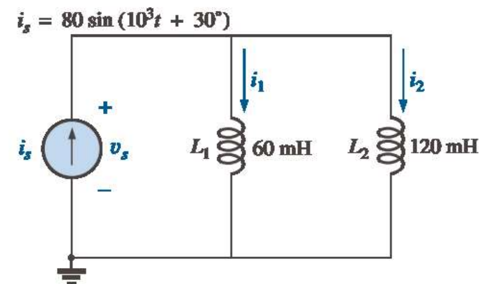


FIG. 14.81
Problem 36.

SECTION 14.8 Conversion between Forms

37. Convert the following from rectangular to polar form:
a. $4 + j6$ **b.** $3 + j3$
c. $5 + j15$ **d.** $500 + j50$
e. $-1000 + j2000$ **f.** $-0.2 + j0.4$

***38.** Convert the following from rectangular to polar form:
a. $-8 - j16$
b. $+8 - j4$
c. $0.02 - j0.003$
d. $-6 \times 10^{-3} - j6 \times 10^{-3}$
e. $200 + j0.02$
f. $-1000 + j20$

39. Convert the following from polar to rectangular form:
a. $6 \angle 40^\circ$ **b.** $12 \angle 120^\circ$
c. $2000 \angle -90^\circ$ **d.** $0.0064 \angle +200^\circ$
e. $48 \angle 2^\circ$ **f.** $5 \times 10^{-4} \angle -20^\circ$

40. Convert the following from polar to rectangular form:
a. $42 \angle 0.15^\circ$
b. $2002 \angle -60^\circ$
c. $0.006 \angle -120^\circ$
d. $8 \times 10^{-3} \angle -220^\circ$
e. $15 \angle +180^\circ$
f. $1.2 \angle -89.9^\circ$

SECTION 14.9 Mathematical Operations with Complex Numbers

41. Perform the following additions in rectangular form:
a. $(4.8 + j7.8) + (4.6 + j0.6)$
b. $(242 + j7) + (3.8 + j44) + (0.4 + j0.7)$
c. $(5 \times 10^{-6} + j75) + (7.4 \times 10^{-7} - j9)$

42. Perform the following subtractions in rectangular form:
a. $(8.8 + j6.2) - (5.6 + j5.6)$
b. $(197 + j243) - (-42.3 - j58)$
c. $(-36.0 + j70) - (-5 - j6) + (10.5 - j72)$

43. Perform the following operations with polar numbers, and leave the answer in polar form:
a. $6 \angle 20^\circ + 8 \angle 80^\circ$
b. $42 \angle 45^\circ + 62 \angle 60^\circ - 70 \angle 120^\circ$
c. $20 \angle -120^\circ - 10 \angle -150^\circ + 8 \angle -210^\circ + 8 \angle +240^\circ$

44. Perform the following multiplications in rectangular form:
a. $(2 + j3)(6 + j8)$
b. $(7.8 + j1)(4 + j2)(7 + j6)$
c. $(400 - j200)(-0.01 - j0.5)(-1 + j3)$

45. Perform the following multiplications in polar form:
a. $(2 \angle 60^\circ)(4 \angle -40^\circ)$
b. $(6.9 \angle 8^\circ)(7.2 \angle -72^\circ)$
c. $(0.002 \angle 120^\circ)(0.5 \angle 200^\circ)(40 \angle + 80^\circ)$

46. Perform the following divisions in polar form:
a. $(42 \angle 10^\circ)/(7 \angle 60^\circ)$
b. $(0.006 \angle 120^\circ)/(30 \angle +60^\circ)$
c. $(4360 \angle -20^\circ)/(40 \angle -210^\circ)$

47. Perform the following divisions, and leave the answer in rectangular form:
a. $(8 + j8)/(2 + j2)$
b. $(8 + j42)/(-6 - j4)$
c. $(-4.5 - j6)/(0.1 - j0.8)$

***48.** Perform the following operations, and express your answer in rectangular form:
a. $\dfrac{(4 + j3) + (6 - j8)}{(3 + j3) - (2 + j3)}$
b. $\dfrac{8 \angle 60^\circ}{(2 \angle 0^\circ) + (100 + j400)}$
c. $\dfrac{(6 \angle 20^\circ)(120 \angle -40^\circ)(3 + j8)}{2 \angle -30^\circ}$

***49.** Perform the following operations, and express your answer in polar form:
a. $\dfrac{(0.4 \angle 60^\circ)^2(300 \angle 40^\circ)}{3 + j9}$
b. $\left(\dfrac{1}{(0.02 \angle 10^\circ)^2}\right)\left(\dfrac{2}{j}\right)^3\left(\dfrac{1}{6^2 - j\sqrt{900}}\right)$

***50.** **a.** Determine a solution for x and y if
$$(x + j5) + (3x + jy) - j6 = 16 \angle 0^\circ$$
b. Determine x if
$$(18 \angle 20^\circ)(x \angle -60^\circ) = 38.64 - j25.72$$

***51.** **a.** Determine a solution for x and y if
$$(5x + j10)(2 - jy) = 90 - j70$$
b. Determine θ if
$$\frac{80 \angle 0^\circ}{20 \angle \theta} = 3.464 - j2$$

SECTION 14.11 Phasors

52. Express the following in phasor form:
a. $\sqrt{2}(180)\sin(\omega t + 40^\circ)$
b. $\sqrt{2}(25 \times 10^{-3})\sin(157t - 60^\circ)$
c. $300 \sin(\omega t - 120^\circ)$

***53.** Express the following in phasor form:
a. $30 \sin(377t - 180^\circ)$
b. $6 \times 10^{-6} \cos \omega t$
c. $5.6 \times 10^{-6} \cos(754t - 40^\circ)$

54. Express the following phasor currents and voltages as sine waves if the frequency is 60 Hz:
a. $\mathbf{I} = 40 \text{ A} \angle 20^\circ$
b. $\mathbf{V} = 120 \text{ V} \angle 10^\circ$
c. $\mathbf{I} = 8 \times 10^{-3}\text{A} \angle -110^\circ$
d. $V = \dfrac{6000}{\sqrt{2}} \text{ V} \angle -180^\circ$

55. For the system in Fig. 14.82, find the sinusoidal expression for the unknown voltage v_a if
$$e_{in} = 60 \sin(377t + 90^\circ)$$
$$v_b = 20 \sin(377t - 45^\circ)$$

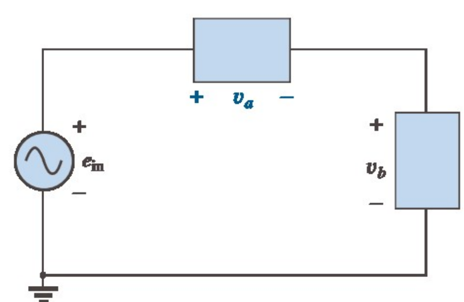


FIG. 14.82
Problem 55.

56. For the system in Fig. 14.83, find the sinusoidal expression for the unknown current i_1 if

$$i_s = 30 \times 10^{-6} \sin(\omega t + 80°)$$
$$i_2 = 4 \times 10^{-6} \sin(\omega t - 50°)$$

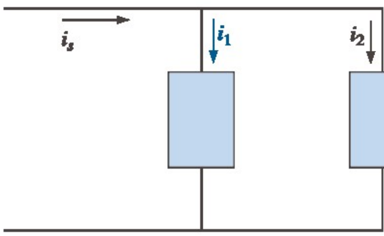


FIG. 14.83
Problem 56.

57. Find the sinusoidal expression for the voltage v_a for the system in Fig. 14.84 if

$$e_{in} = 120 \sin(\omega t + 30°)$$
$$v_b = 30 \sin(\omega t + 60°)$$
$$v_c = 40 \sin(\omega t - 90°)$$

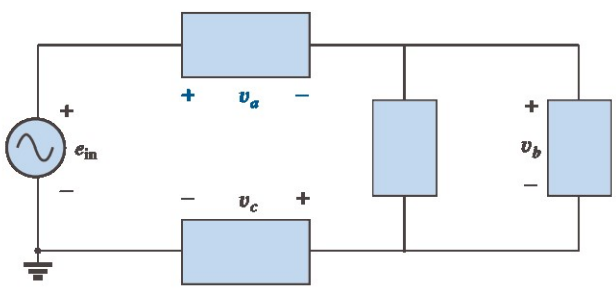


FIG. 14.84
Problem 57.

***58.** Find the sinusoidal expression for the current i_1 for the system in Fig. 14.85 if

$$i_s = 18 \times 10^{-3} \sin(377t + 180°)$$
$$i_2 = 8 \times 10^{-3} \sin(377t + 90°)$$
$$i_3 = 2i_2$$

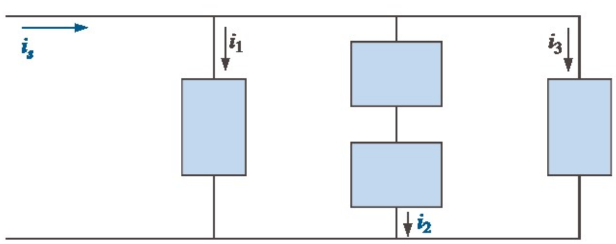


FIG. 14.85
Problem 58.

SECTION 14.12 Computer Analysis

PSpice or Multisim

59. Plot i_c and v_c versus time for the network in Fig. 14.74 for two cycles if the frequency is 0.2 kHz.

60. Plot the magnitude and phase angle of the current i_c versus frequency (100 Hz to 100 kHz) for the network in Fig. 14.74.

***61.** Plot the total impedance of the configuration in Fig. 14.24(a) versus frequency (100 kHz to 100 MHz) for the following parameter values: $C = 0.1\ \mu F$, $L_s = 0.2\ \mu H$, $R_s = 2 M\Omega$, and $R_p = 100\ M\Omega$. For what frequency range is the capacitor "capacitive"?

GLOSSARY

Average or real power The power delivered to and dissipated by the load over a full cycle.

Complex conjugate A complex number defined by simply changing the sign of an imaginary component of a complex number in the rectangular form.

Complex number A number that represents a point in a two-dimensional plane located with reference to two distinct axes. It defines a vector drawn from the origin to that point.

Derivative The instantaneous rate of change of a function with respect to time or another variable.

Leading and lagging power factors An indication of whether a network is primarily capacitive or inductive in nature. Leading power factors are associated with capacitive networks and lagging power factors with inductive networks.

Phasor A radius vector that has a constant magnitude at a fixed angle from the positive real axis and that represents a sinusoidal voltage or current in the vector domain.

Phasor diagram A "snapshot" of the phasors that represent a number of sinusoidal waveforms at $t = 0$.

Polar form A method of defining a point in a complex plane that includes a single magnitude to represent the distance from the origin and an angle to reflect the counterclockwise distance from the positive real axis.

Power factor (F_p) An indication of how reactive or resistive an electrical system is. The higher the power factor, the greater is the resistive component.

Reactance The opposition of an inductor or a capacitor to the flow of charge that results in the continual exchange of energy between the circuit and magnetic field of an inductor or the electric field of a capacitor.

Reciprocal A format defined by 1 divided by the complex number.

Rectangular form A method of defining a point in a complex plane that includes the magnitude of the real component and the magnitude of the imaginary component, the latter component being defined by an associated letter *j*.