71–72 Describe how the graph of g is obtained from the graph of f.

71. (a) $g(x, y) = f(x, y) + 2$
(b) $g(x, y) = 2f(x, y)$
(c) $g(x, y) = -f(x, y)$
(d) $g(x, y) = 2 - f(x, y)$

72. (a) $g(x, y) = f(x - 2, y)$
(b) $g(x, y) = f(x, y + 2)$
(c) $g(x, y) = f(x + 3, y - 4)$

73–74 Graph the function using various domains and viewpoints that give good views of the "peaks and valleys." Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be "local maximum points"? What about "local minimum points"?

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$

74. $f(x, y) = xye^{-x^2-y^2}$

75–76 Graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both x and y become large? What happens as (x, y) approaches the origin?

75. $f(x, y) = \dfrac{x + y}{x^2 + y^2}$ **76.** $f(x, y) = \dfrac{xy}{x^2 + y^2}$

---

77. Investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c?

78. Investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2-y^2}$$

How does the shape of the graph depend on the numbers a and b?

79. Investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

80. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = e^{\sqrt{x^2+y^2}}$$

$$f(x, y) = \ln\sqrt{x^2 + y^2}$$

$$f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$$

and $$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$$

obtained from the graph of g?

81. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bL^{\alpha}K^{1-\alpha}$ can be expressed as

$$\ln \frac{P}{K} = \ln b + \alpha \ln \frac{L}{K}$$

(b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 4) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.

(c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

14.2 Limits and Continuity

Limits of Functions of Two Variables

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

$\lim_{x \to 1} x^2+1$

$x=1$ $1{,}001$

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

Table 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any particular number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ does not exist}$$

$\frac{\sin(0)}{0} = \frac{0}{0}$

Cal 1

$\frac{0}{0}, \frac{\infty}{\infty}$

$\Rightarrow$ L.H

Cal 3

on 2N, 3V

đâu) nữa 1)

L.H

In general, we use the notation

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) (staying within the domain of f). In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

1 Definitio Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small, but not 0. (Compare to the definition of a limit for a function of a single variable, Definition 2.4.2.) Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

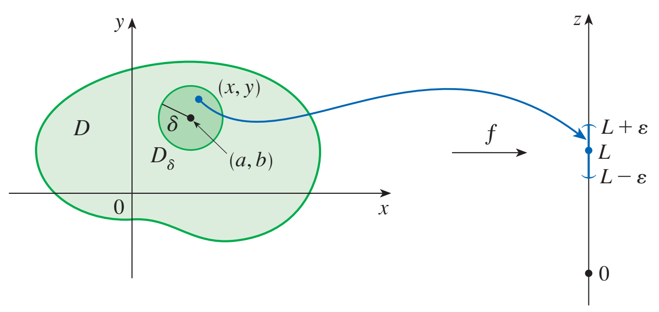

FIGURE 1

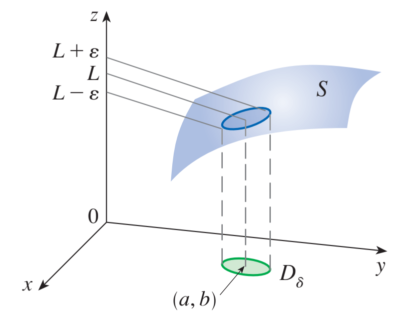

FIGURE 2

Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

Showing That a Limit Does Not Exist

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right. We recall from Chapter 2 that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

For functions of two variables, the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit *no matter how* (x, y) approaches (a, b). Thus one way to show that $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist is to find different paths of approach along which the function has different limits.

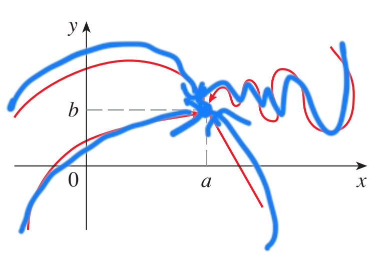

FIGURE 3
Different paths approaching (a, b)

> If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

EXAMPLE 1 Show that $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. On this path $y = 0$ for every point (x, y), so the function becomes $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$ and thus

$$f(x, y) \to 1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

Ex ①

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

2N

3V

path

$$\lim_{x\to +} \neq \lim_{x\to -}$$

[1] $y=0$ $\quad \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}} = \boxed{1}$ ✓

[2] $x=0$ $\quad \lim_{y\to 0} \frac{-\cancel{y^2}}{\cancel{y^2}} = \boxed{-1}$

[3] $y=x$ $\quad \lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2} = \frac{0}{2x^2} = \boxed{0}$

$\Rightarrow$ lim Does not exist

D.N.E

الهدف منه، لمسارات path حصلوا اتنتين قيمة D.N.E

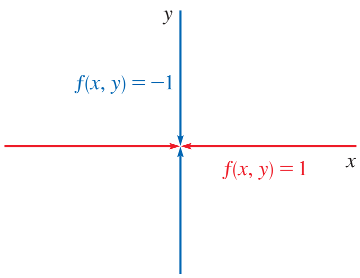

FIGURE 4

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

(See Figure 4.) Since f has two different limits as (x, y) approaches $(0, 0)$ along two different lines, the given limit does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.) ■

EXAMPLE 2 If $f(x, y) = \dfrac{xy}{x^2 + y^2}$, does $\displaystyle\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the two axes, that does *not* show that the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\qquad f(x, y) \to \frac{1}{2} \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along } y = x$

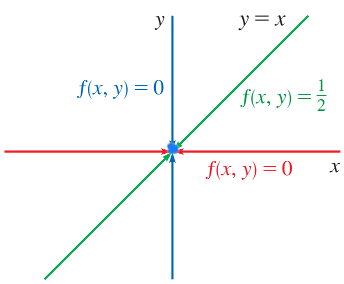

FIGURE 5

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. ■

Figure 6 sheds some light on Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

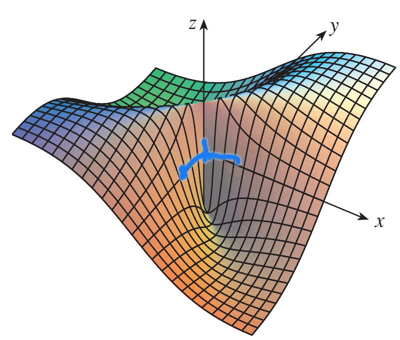

FIGURE 6

$f(x, y) = \dfrac{xy}{x^2 + y^2}$

EXAMPLE 3 If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\displaystyle\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any line through the origin. If the line is not the y-axis, then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

Ex ②

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$\boxed{1}$ $y=0 \Longrightarrow \lim_{x\to 0} \frac{0x}{x^2+0^2} = \frac{0}{x^2} = \boxed{0}$

$\boxed{2}$ $x=0 \Longrightarrow \lim_{y\to 0} \frac{0y}{0^2+y^2} = \frac{0}{y^2} = \boxed{0}$

$\boxed{3}$ $y=x \Longrightarrow \lim_{x\to 0} \frac{xx}{x^2+x^2} = \frac{\cancel{x^2}}{2\cancel{x^2}} = \boxed{\frac{1}{2}}$

$$0 \neq \frac{1}{2}$$

$\Longrightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E

اصعب نقطة ونحن نألف مسار جديد يحقق
نقطة النهاية

Ex 3

$$f(x,y) = \frac{xy^2}{x^2+y^4} \qquad \lim_{(x,y)\to(0,0)} f(x,y)$$

1) $y=0$

$$\lim_{x\to 0} \frac{x(0)^2}{x^2+0^4} = \frac{0}{x^2} = \boxed{0}$$

2) $x=y^2$

$$\lim_{y\to 0} \frac{y^2 \cdot y^2}{(y^2)^2+y^4} = \frac{1\cancel{y^4}}{2\cancel{y^4}} = \boxed{\frac{1}{2}}$$

$(y^2)^2 = y^4$

$$0 \neq \frac{1}{2} \implies \text{D.N.E}$$

3) $y=x$

$$\lim_{x\to 0} \frac{x(x^2)}{x^2+x^4} = \frac{x^3}{x^2+x^4} = \frac{0}{0+0} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{x^3}{\cancel{x^2}(1+x^2)} = \frac{x}{1+x^2} = \frac{0}{1+0} = \frac{0}{1} = 0$$

Figure 7 shows the graph of the function in Example 3. Notice the ridge above the parabola $x = y^2$.

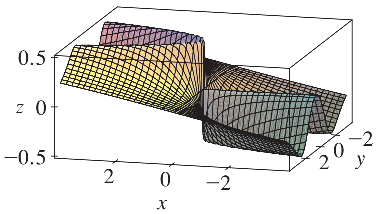

FIGURE 7

So $$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } y = mx$$

We get the same result as $(x, y) \to (0, 0)$ along the line $x = 0$. Thus f has the same limiting value along every line through the origin. But that does not show that the given limit is 0, for if we now let $(x, y) \to (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $$f(x, y) \to \tfrac{1}{2} \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } x = y^2$$

Since different paths lead to different limiting values, the given limit does not exist. ■

Properties of Limits

Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 2.3 can be extended to functions of two variables. Assuming that the indicated limits exist, we can state these laws verbally as follows:

Sum Law 1. The limit of a sum is the sum of the limits.

Difference Law 2. The limit of a difference is the difference of the limits.

Constant Multiple Law 3. The limit of a constant times a function is the constant times the limit of the function.

Product Law 4. The limit of a product is the product of the limits.

Quotient Law 5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

In Exercise 54, you are asked to prove the following special limits:

$$\boxed{2} \qquad \lim_{(x, y) \to (a, b)} x = a \qquad \lim_{(x, y) \to (a, b)} y = b \qquad \lim_{(x, y) \to (a, b)} c = c$$

A **polynomial function** of two variables (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of two polynomials. For instance,

$$p(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$q(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The special limits in (2) along with the limit laws allow us to evaluate the limit of any polynomial function p by direct substitution:

$$\boxed{3} \qquad \lim_{(x, y) \to (a, b)} p(x, y) = p(a, b)$$

Similarly, for any rational function $q(x, y) = p(x, y)/r(x, y)$ we have

$$\boxed{4} \qquad \lim_{(x, y) \to (a, b)} q(x, y) = \lim_{(x, y) \to (a, b)} \frac{p(x, y)}{r(x, y)} = \frac{p(a, b)}{r(a, b)} = q(a, b)$$

provided that (a, b) is in the domain of q.

EXAMPLE 4 Evaluate $\displaystyle\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11 \quad \blacksquare$$

EXAMPLE 5 Evaluate $\displaystyle\lim_{(x,y)\to(-2,3)} \frac{x^2y + 1}{x^3y^2 - 2x}$ if it exists.

SOLUTION The function $f(x, y) = (x^2y + 1)/(x^3y^2 - 2x)$ is a rational function and the point $(-2, 3)$ is in its domain (the denominator is not 0 there), so we can evaluate the limit by direct substitution:

$$\lim_{(x,y)\to(-2,3)} \frac{x^2y + 1}{x^3y^2 - 2x} = \frac{(-2)^2(3) + 1}{(-2)^3(3)^2 - 2(-2)} = -\frac{13}{68} \quad \blacksquare$$

The Squeeze Theorem also holds for functions of two or more variables. In the next example we find a limit in two different ways: by using the definition of limit and by using the Squeeze Theorem.

EXAMPLE 6 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

SOLUTION 1 As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

that is,

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \frac{3x^2|y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2 + y^2) \leqslant 1$ and therefore

$$\boxed{5} \qquad \frac{3x^2|y|}{x^2 + y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then by (5) we have

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leqslant 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition 1,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0$$

Ex 4

$$\lim_{(x,y)\to(1,2)} x^2y^3 - x^3y^2 + 3x + 2y$$

$$= 1(2)^3 - 1(2)^2 + 3(1) + 2(2)$$

$$= \underline{\underline{8}} \cancel{-4} + \underline{\underline{3}} \cancel{+4} = \boxed{11}$$

Ex 5

$$\lim_{x,y\to(-2,3)} \frac{x^2y+1}{x^3y^2-2x}$$

$\frac{\infty}{\infty}$ $\frac{0}{0}$

path

او اشوف طريقة

حل اخرى

4 2 12

$$\frac{(-2)(3)+1}{(-2)^3(3)^2-2(-2)} = \boxed{\frac{13}{-68}}$$

$(-8)(9)+4$

$-72+4$

Ex 6

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \frac{0}{0+0} = \boxed{\frac{0}{0}}$$

3 2

Path

$x=0 \longrightarrow \lim_{y\to 0} \frac{0}{0+y^2} = \boxed{0}$

$y=0 \longrightarrow \lim_{x\to 0} \frac{0}{x^2+0} = \boxed{0}$

$y=x \longrightarrow \lim_{x\to 0} \frac{3x^2x}{x^2+x^2} = \frac{3x^3}{2x^2} = \frac{3}{2}x = \boxed{0}$

$x=y^2 \longrightarrow \lim_{y\to 0} \frac{3(y^2)^2y^1}{(y^2)^2+y^2} = \frac{3y^5}{y^4+y^2}$

$$\frac{3y^{\cancel{5}\,3}}{\cancel{y^2}(y^2+1)} = \frac{3y^3}{y^2+1} = \frac{0}{0+1} = \frac{0}{1} = \boxed{0}$$

Squeeze theorem

$$\boxed{\frac{x^2y+1}{x^5y+\ldots}} \longrightarrow \frac{1}{y}$$

$$\left| \frac{+3x^2y}{x^2+y^2} \right| < \varepsilon$$

$|-5| \longrightarrow +5$ $|+7| \longrightarrow +7$

$$\frac{3x^2}{x^2\cancel{+y^2}} |y| < \varepsilon$$

$$\frac{30}{42} = \frac{6\times 5}{6\times 7} \leq \frac{5\times 7}{6\times 7} = \frac{35}{42}$$

$$\frac{3x^2|y|}{x^2+y^2} \leq \frac{3\cancel{x^2}|y|}{\cancel{x^2}}$$

$$-3|y| \leq \frac{3x^2|y|}{x^2+y^2} \leq +3|y|$$

$$0 \quad -3|y| \leq \frac{3x^2y}{x^2+y^2} \leq 3|y|$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad \boxed{0} \qquad 0$$

SOLUTION 2 As in Solution 1,

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \leqslant 3|y|$$

so

$$-3|y| \leqslant \frac{3x^2y}{x^2+y^2} \leqslant 3|y|$$

Now $|y| \to 0$ as $y \to 0$ so $\lim_{(x,y)\to(0,0)} (-3|y|) = 0$ and $\lim_{(x,y)\to(0,0)} (3|y|) = 0$ (using Limit Law 3). Thus, by the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x\to a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

6 Definitio A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say that f is **continuous on** D if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

We have already seen that limits of polynomial functions can be evaluated by direct substitution (Equation 3). It follows by the definition of continuity that *all polynomials are continuous on* $\mathbb{R}^2$. Likewise, Equation 4 shows that *any rational function is continuous on its domain*. In general, using properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

EXAMPLE 7 Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$.

EXAMPLE 8 Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

D.N.E

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist (see Example 1).

Cont

Ex # $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

cont everywhere unless
$(x,y) \to (0,0)$

cont

$\lim_{x \to +} \; = \; \lim_{x \to -}$

Figure 8 shows the graph of the continuous function in Example 9.

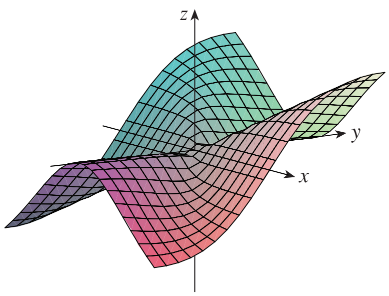

FIGURE 8

EXAMPLE 9 Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 6, we have

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on $\mathbb{R}^2$. ■

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

e cont every where

EXAMPLE 10 Where is the function $h(x, y) = e^{-(x^2+y^2)}$ continuous?

SOLUTION The function $f(x, y) = x^2 + y^2$ is a polynomial and thus is continuous on $\mathbb{R}^2$. Because the function $g(t) = e^{-t}$ is continuous for all values of t, the composite function

$$h(x, y) = g(f(x, y)) = e^{-(x^2+y^2)}$$

is continuous on $\mathbb{R}^2$. The function h is graphed in Figure 9.

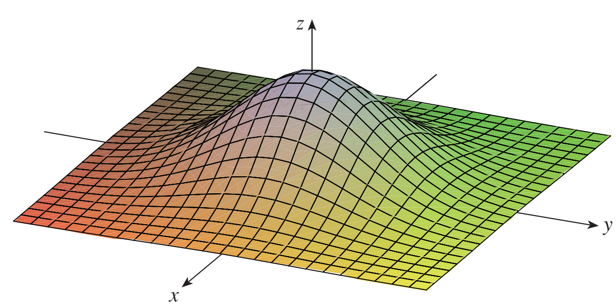

FIGURE 9 The function $h(x, y) = e^{-(x^2+y^2)}$ is continuous everywhere. ■

EXAMPLE 11 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

Ex 11 $h(x,y) = \arctan\left(\frac{y}{x}\right)$

$\arctan = \tan^{-1}$

$h(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

cont every where unless

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$\boxed{x = 0}$

is continuous except where $x = 0$. The graph in Figure 10 shows the break in the graph of h above the y-axis.

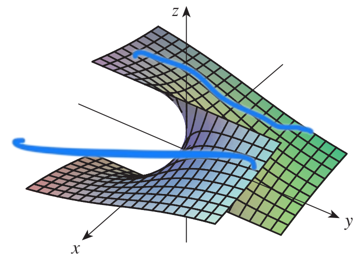

FIGURE 10
The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) (staying within the domain of f). Because the distance between two points (x, y, z) and (a, b, c) in $\mathbb{R}^3$ is given by $\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$, we can write the precise definition as follows: for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y, z) \text{ is in the domain of } f \text{ and } 0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$$
$$\text{then } |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 14.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

7 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (7) is just the definition of a limit for functions of a single variable (Definition 2.4.2). For the case $n = 2$, we have $\mathbf{x} = \langle x, y \rangle$, $\mathbf{a} = \langle a, b \rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x - a)^2 + (y - b)^2}$, so (7) becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a, b, c \rangle$, and (7) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

14.2 Exercises

1. Suppose that $\lim_{(x,y)\to(3,1)} f(x, y) = 6$. What can you say about the value of $f(3, 1)$? What if f is continuous?

2. Explain why each function is continuous or discontinuous.
(a) The outdoor temperature as a function of longitude, latitude, and time
(b) Elevation (height above sea level) as a function of longitude, latitude, and time
(c) The cost of a taxi ride as a function of distance traveled and time

3–4 Use a table of numerical values of $f(x, y)$ for (x, y) near the origin to make a conjecture about the value of the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. Then explain why your guess is correct.

3. $f(x, y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ **4.** $f(x, y) = \dfrac{2xy}{x^2 + 2y^2}$

5–12 Find the limit.

5. $\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$

6. $\lim_{(x,y)\to(5,-2)} (x^2y + 3xy^2 + 4)$

7. $\lim_{(x,y)\to(-3,1)} \dfrac{x^2y - xy^3}{x - y + 2}$ **8.** $\lim_{(x,y)\to(2,-1)} \dfrac{x^2y + xy^2}{x^2 - y^2}$

9. $\lim_{(x,y)\to(\pi,\pi/2)} y \sin(x - y)$ **10.** $\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}}$

11. $\lim_{(x,y)\to(1,1)} \left(\dfrac{x^2y^3 - x^3y^2}{x^2 - y^2}\right)$ **12.** $\lim_{(x,y)\to(\pi,\pi/2)} \dfrac{\cos y - \sin 2y}{\cos x \cos y}$

13–18 Show that the limit does not exist.

13. $\lim_{(x,y)\to(0,0)} \dfrac{y^2}{x^2 + y^2}$ **14.** $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + 3y^2}$

15. $\lim_{(x,y)\to(0,0)} \dfrac{(x + y)^2}{x^2 + y^2}$ **16.** $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + xy^2}{x^4 + y^2}$

17. $\lim_{(x,y)\to(0,0)} \dfrac{y^2 \sin^2 x}{x^4 + y^4}$ **18.** $\lim_{(x,y)\to(1,1)} \dfrac{y - x}{1 - y + \ln x}$

19–30 Find the limit, if it exists, or show that the limit does not exist.

19. $\lim_{(x,y)\to(-1,-2)} (x^2y - xy^2 + 3)^3$

20. $\lim_{(x,y)\to(\pi,1/2)} e^{xy} \sin xy$

21. $\lim_{(x,y)\to(2,3)} \dfrac{3x - 2y}{4x^2 - y^2}$ **22.** $\lim_{(x,y)\to(1,2)} \dfrac{2x - y}{4x^2 - y^2}$

23. $\lim_{(x,y)\to(0,0)} \dfrac{xy^2 \cos y}{x^2 + y^4}$ **24.** $\lim_{(x,y)\to(0,0)} \dfrac{x^3 - y^3}{x^2 + xy + y^2}$

25. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

26. $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^2 + y^8}$

27. $\lim_{(x,y,z)\to(6,1,-2)} \sqrt{x + z} \cos(\pi y)$

28. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz}{x^2 + y^2 + z^2}$

29. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

30. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^4 + y^2 + z^3}{x^4 + 2y^2 + z}$

31–34 Use the Squeeze Theorem to find the limit.

31. $\lim_{(x,y)\to(0,0)} xy \sin \dfrac{1}{x^2 + y^2}$ **32.** $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}}$

33. $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^4 + y^4}$

34. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^2 + y^2 + z^2}$

35–36 Use a graph of the function to explain why the limit does not exist.

35. $\lim_{(x,y)\to(0,0)} \dfrac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$ **36.** $\lim_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2 + y^6}$

37–38 Find $h(x, y) = g(f(x, y))$ and the set of points at which h is continuous.

37. $g(t) = t^2 + \sqrt{t}$, $f(x, y) = 2x + 3y - 6$

38. $g(t) = t + \ln t$, $f(x, y) = \dfrac{1 - xy}{1 + x^2y^2}$

39–40 Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

39. $f(x, y) = e^{1/(x-y)}$ **40.** $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$

41–50 Determine the set of points at which the function is continuous.

41. $F(x, y) = \dfrac{xy}{1 + e^{x-y}}$ **42.** $F(x, y) = \cos\sqrt{1 + x - y}$

43. $F(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$ **44.** $H(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

45. $G(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2}$

46. $G(x, y) = \ln(1 + x - y)$

47. $f(x, y, z) = \arcsin(x^2 + y^2 + z^2)$

48. $f(x, y, z) = \sqrt{y - x^2} \ln z$

49. $f(x, y) = \begin{cases} \dfrac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

50. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

51–53 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geqslant 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

51. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2}$

52. $\displaystyle\lim_{(x, y) \to (0, 0)} (x^2 + y^2) \ln(x^2 + y^2)$

53. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{e^{-x^2 - y^2} - 1}{x^2 + y^2}$

54. Prove the three special limits in (2).

55. At the beginning of this section we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed on the basis of numerical evidence that $f(x, y) \to 1$ as $(x, y) \to (0, 0)$. Use polar coordinates to confirm the value of the limit. Then graph the function.

56. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

57. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leqslant 0 \quad \text{or} \quad y \geqslant x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

(a) Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $0 < a < 4$.
(b) Despite part (a), show that f is discontinuous at $(0, 0)$.
(c) Show that f is discontinuous on two entire curves.

58. Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$. [*Hint:* Consider $|\mathbf{x} - \mathbf{a}|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

59. If $\mathbf{c} \in V_n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on $\mathbb{R}^n$.

14.3 Partial Derivatives

Partial Derivatives of Functions of Two Variables

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

Table 1 Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°C)	T \ H	40	45	50	55	60	65	70	75	80
	26	28	28	29	31	31	32	33	34	35
	28	31	32	33	34	35	36	37	38	39
	30	34	35	36	37	38	40	41	42	43
	32	37	38	39	41	42	43	45	46	47
	34	41	42	43	45	47	48	49	51	52
	36	43	45	47	48	50	51	53	54	56