



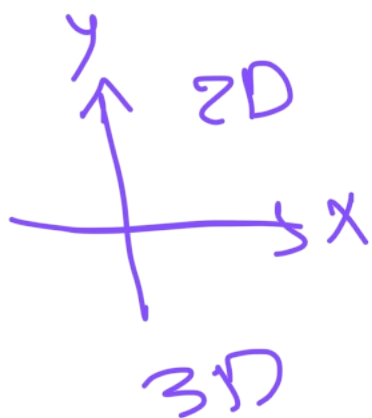
A function of two variables can describe the shape of a surface like the one formed by these sand dunes. In Exercise 14.6.40 you are asked to use partial derivatives to compute the rate of change of elevation as a hiker walks in different directions.

Seppfriedhuber / E+ / Getty Images

14

Partial Derivatives

SO FAR WE HAVE DEALT with the calculus of functions of a single variable. But, in the real world, physical quantities often depend on two or more variables, so in this chapter we turn our attention to functions of several variables and extend the basic ideas of differential calculus to such functions.



cal 1,2

$$f(x) = \sin x \quad \checkmark$$

$$y = \sqrt{x} \quad \checkmark$$

⋮

cal 3

$$f(x, y) = \sin x + \sqrt{y}$$

$$f(x, y, z) = \sin x z - \ln y$$

$$f(x) = x^2$$

indep $\rightarrow x$
 dep $f(x)$

$$\begin{aligned}
 & \cdot x=1 \\
 & \cdot x=2 \\
 & \cdot x=3
 \end{aligned}$$

$$\begin{aligned}
 x=1 & \rightarrow f(1) = 1^2 = 1 \\
 x=2 & \rightarrow f(2) = 2^2 = 4
 \end{aligned}$$

14.1

Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

Functions of Two Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y or as a function of the pair (x, y) . We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h . In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h , and we can write $V(r, h) = \pi r^2 h$.

Definition A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y) . The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

A function of two variables is just a function whose domain is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$. One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where the domain D is represented as a subset of the xy -plane and the range is a set of numbers on a real line, shown as a z -axis. For instance, if $f(x, y)$ represents the temperature at a point (x, y) in a flat metal plate with the shape of D , we can think of the z -axis as a thermometer displaying the recorded temperatures.

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression defines a real number.

EXAMPLE 1 For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a) $f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$

(b) $f(x, y) = x \ln(y^2 - x)$

SOLUTION

(a) $f(3, 2) = \frac{\sqrt{3 + 2 + 1}}{3 - 1} = \frac{\sqrt{6}}{2}$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) \mid x + y + 1 \geq 0, x \neq 1\}$$

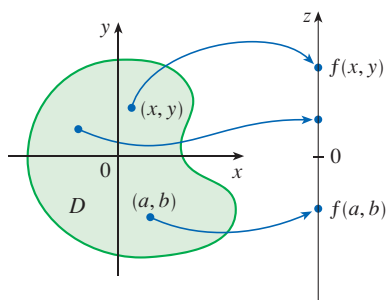


FIGURE 1

Ex 1

$$a) f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$$

$$f(3,2) = \frac{\sqrt{3+2+1}}{3-1}$$

$$= \frac{\sqrt{6}}{2}$$

ind

dep

$$b) f(x,y) = x \ln(y^2 - x) \quad f(3,2)$$

$$\begin{aligned} f(3,2) &= 3 \ln(2^2 - 3) \\ &= 3 \ln(4 - 3) = 3 \ln 1 \\ &= 0 \end{aligned}$$

Domain

→

المجال

التي تسمح لنا بالتعبير.

$$f(x) = x + 1 \rightarrow \text{Dom} = \mathbb{R}$$

$$f(x) = \frac{1}{x} \rightarrow \text{Dom} = \mathbb{R} - \{0\}$$

$$f(x) = \sqrt{x-3} \rightarrow x-3 \geq 0$$

~~$\sqrt{x}$~~

~~$\sqrt{x}$~~

$$x \geq 3$$

$$\text{Dom} = [3, \infty)$$

مجال

$$f(x) = \frac{1}{\sqrt{x-3}} \rightarrow \text{Dom} = (3, \infty)$$

مجال لا يسمح

* Domain

لأبداً دالة $\mathbb{R}$ $\mathbb{R}^2$ $\mathbb{R}^3$
 $2D$ $3D$ $4D$

سأذكر - $\frac{\text{سواء}}{\text{مفاد} \neq 0} = \text{أشكال، كلام}$ [1]

سأبدأ بالجزء $\sqrt[n]{\text{ن}} \geq 0$ [2]
 الزوجي الأكبر من
 أو يساوي صفر

سأبدأ بـ $\ln(\text{ن}) > 0$ [3]
 الأكبر من صفر

Ex ①

$$a) f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$$

Domain

$$\textcircled{1} \quad x \neq 1$$

$$\textcircled{2} \quad x+y+1 \geq 0$$

$$D = \{(x,y) \mid x+y+1 \geq 0, x \neq 1\}$$

$$b) f(x,y) = x \ln(y^2 - x)$$

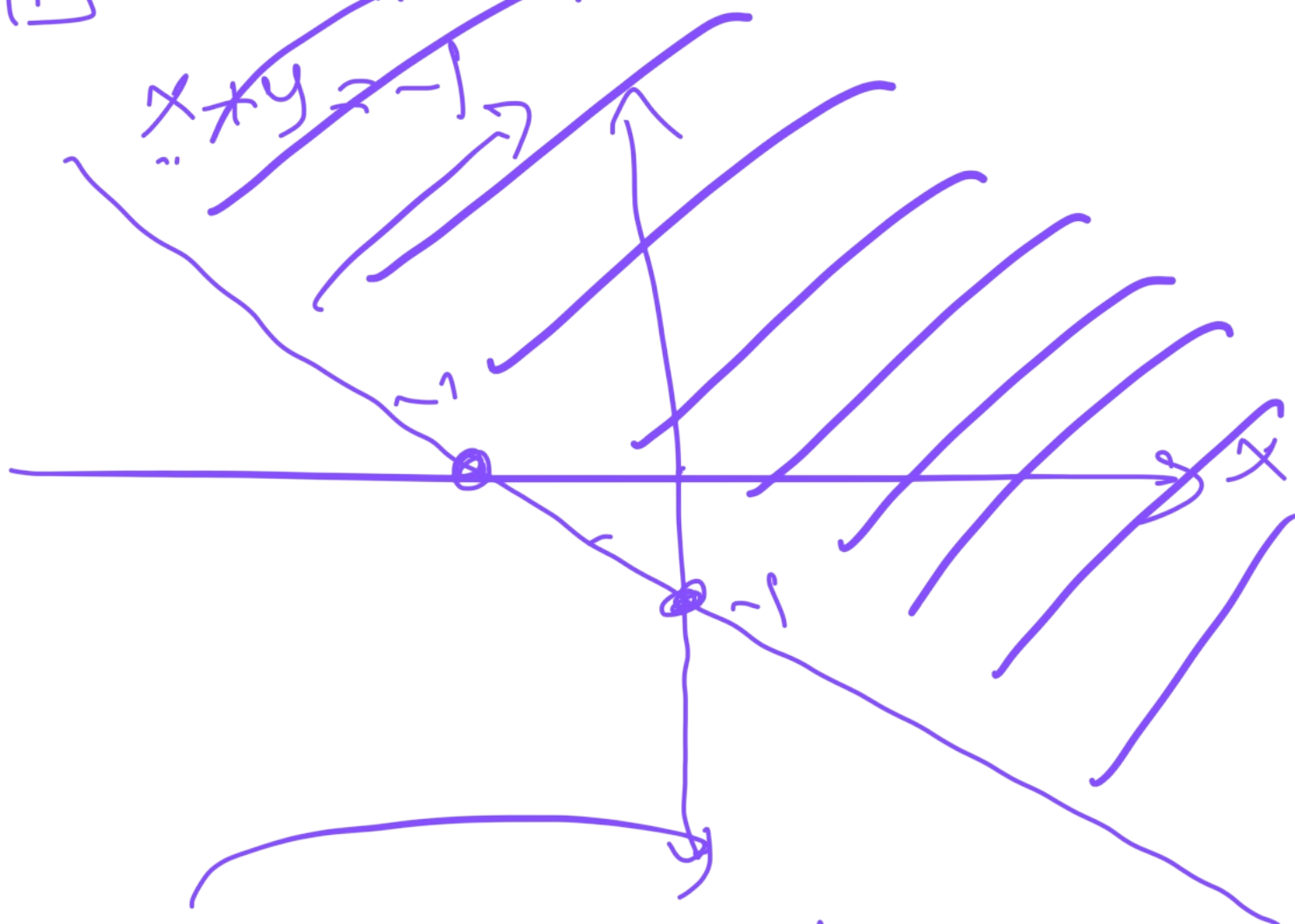
$$\text{Dom} \quad y^2 - x > 0 \rightarrow y^2 > x$$

$$D = \{(x,y) \mid \underline{y^2} > x\}$$

1

$$x + y + 1 = 0$$

$$x * y = -1$$



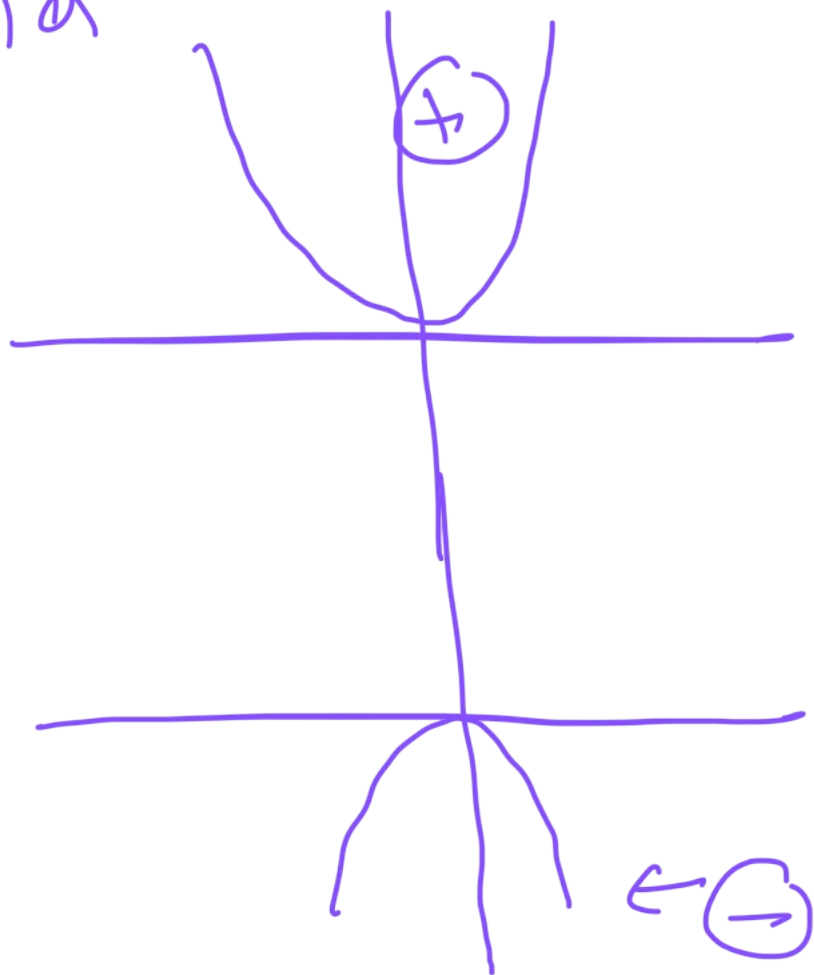
$$x = 0 \rightarrow y = -1$$

$$y = 0 \rightarrow x = -1$$



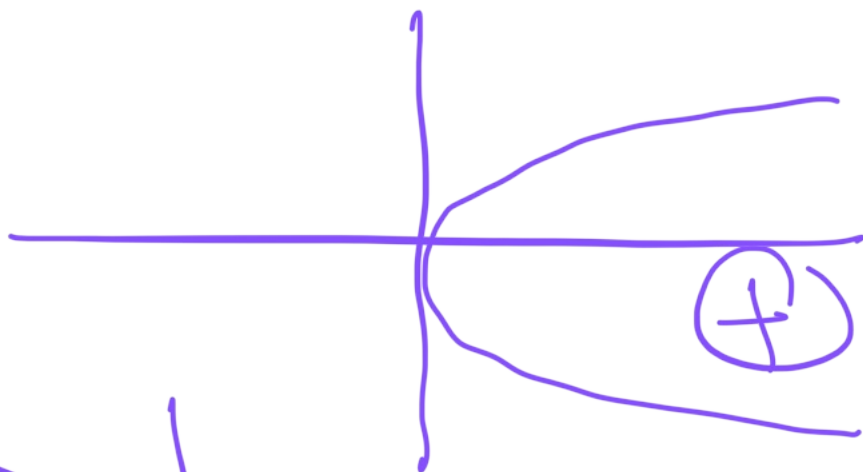
2) parabola

$$y' = x^2$$

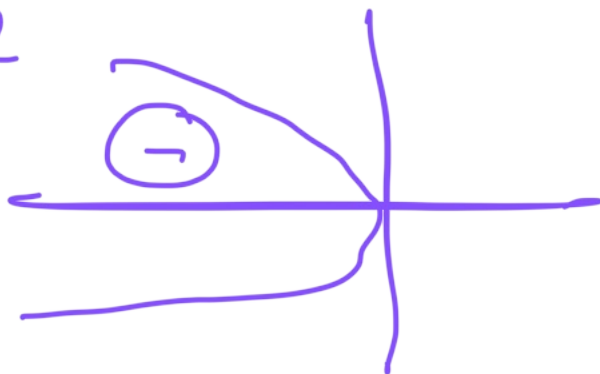


$$y = \ominus x^2$$

$$x' = y^2$$



$$x = \ominus y^2$$



The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain (see Figure 2).

(b) $f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is $D = \{(x, y) \mid x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$. (See Figure 3.)

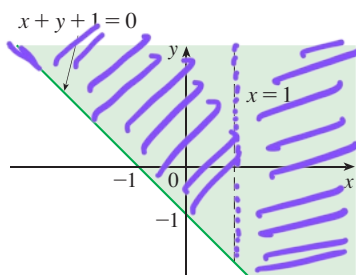


FIGURE 2

Domain of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$

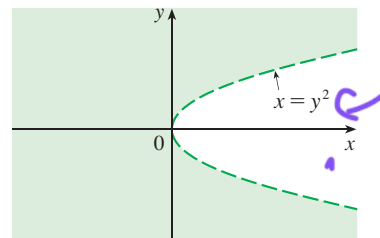


FIGURE 3

Domain of $f(x, y) = x \ln(y^2 - x)$

EXAMPLE 2 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

which is the disk with center $(0, 0)$ and radius 3. (See Figure 4.) The range of g is

$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

Since z is a positive square root, $z \geq 0$. Also, because $9 - x^2 - y^2 \leq 9$, we have

$$\sqrt{9 - x^2 - y^2} \leq 3$$

So the range is

$$\{z \mid 0 \leq z \leq 3\} = [0, 3]$$

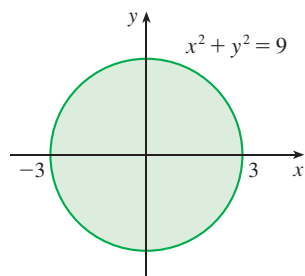


FIGURE 4

Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Not all functions can be represented by explicit formulas. The function in the next example is described verbally and by numerical estimates of its values.

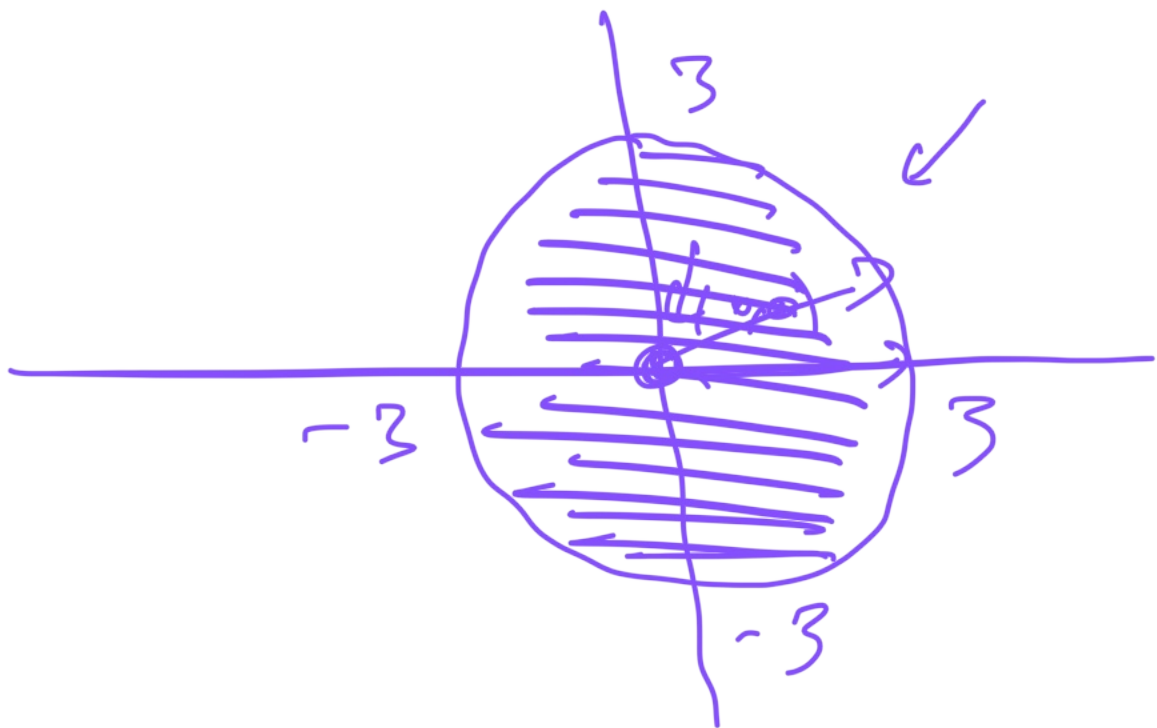
EXAMPLE 3 In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v . So W is a function of

Ex ② $g(x,y) = \sqrt{a - x^2 - y^2}$

Domain $a - x^2 - y^2 \geq 0$

$\rightarrow x^2 + y^2 \leq a$

$x^2 + y^2 = a \rightarrow$ Circle Eq
center $(0,0)$ radius $= \sqrt{a} = 3$



Range : $\longrightarrow$

or,

$$x=1 \longrightarrow y=3$$

Range

$$0 \leq z \leq 3$$

$$\underline{g(x,y)} = \sqrt{a - x^2 - y^2}$$

$$(z)^2 = \left(\sqrt{a - x^2 - y^2} \right)^2$$

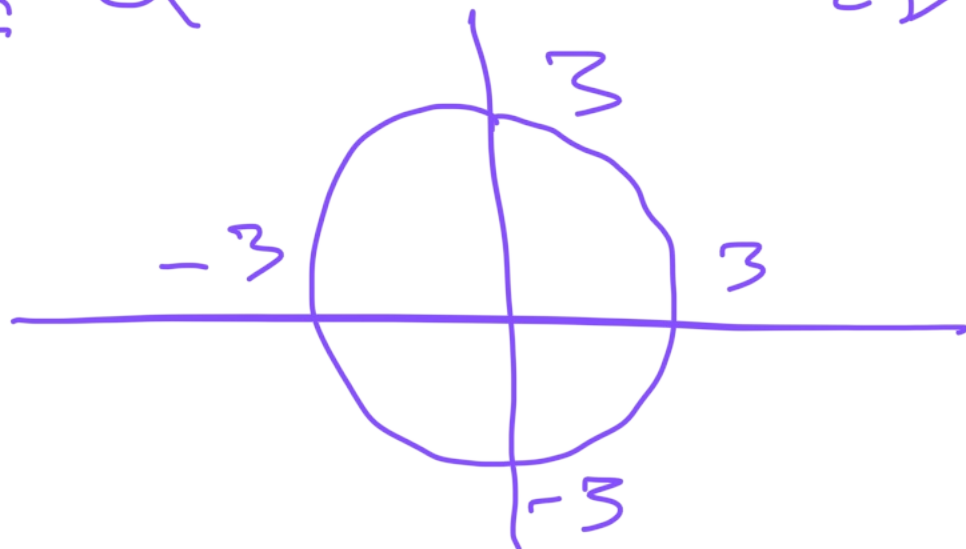
$$z^2 = a - x^2 - y^2$$

$$\Rightarrow \boxed{x^2 + y^2 + z^2 = a}$$

2) Equation of sphere
center $(0,0,0)$ radius $= \underline{\underline{\sqrt{a}=3}}$

$$\underline{\underline{x^2 + y^2 = a}}$$

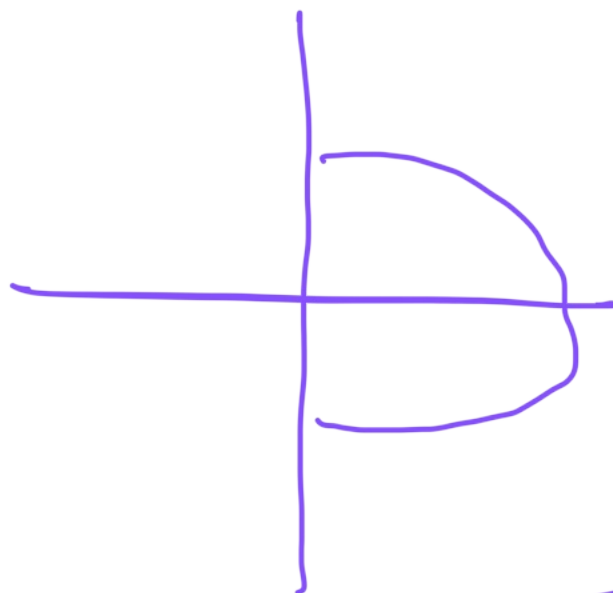
0.1)



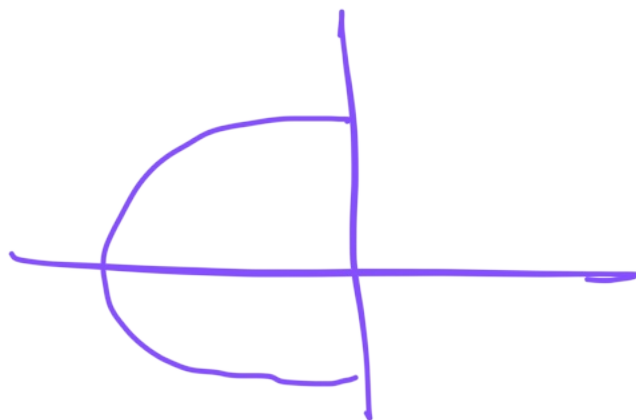
2D

$$\sqrt{x^2} = \sqrt{a - y^2}$$

$$x' = +\sqrt{a - y^2}$$

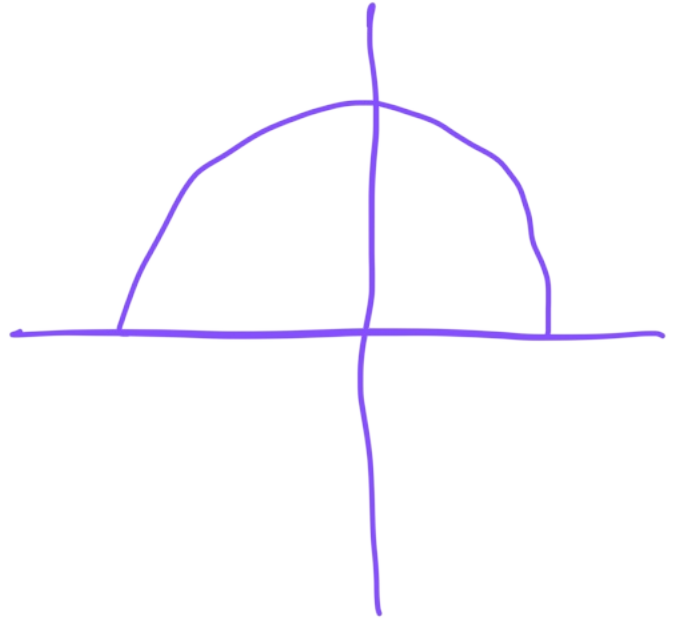


$$x = \ominus \sqrt{a - y^2}$$

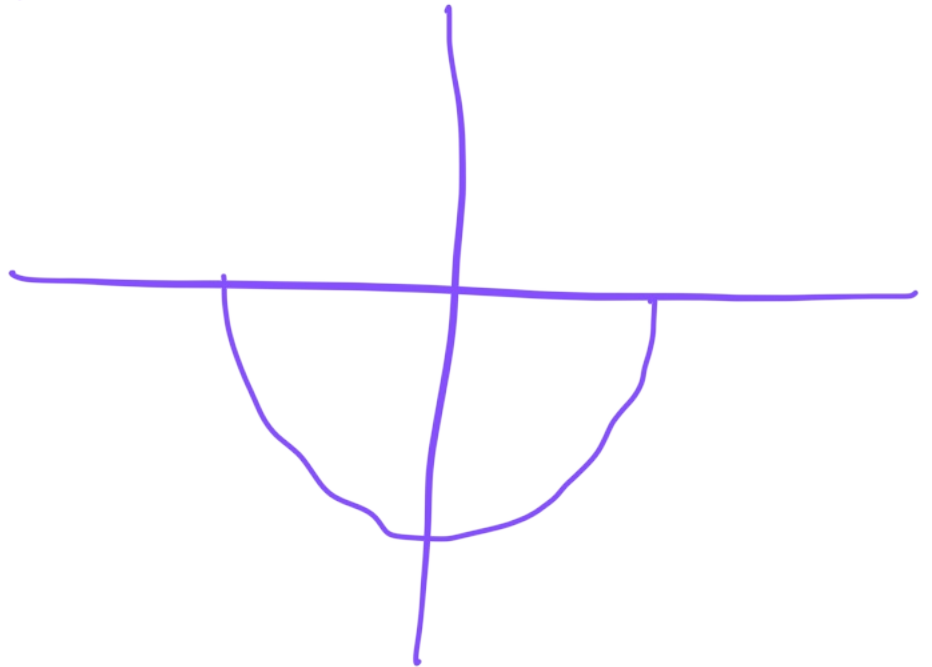


$$\sqrt{y^2} = \sqrt{a - x^2}$$

$$\Rightarrow y' = +\sqrt{a - x^2}$$



$$y = -\sqrt{a - x^2}$$



The Wind-Chill Index

The wind-chill index measures how cold it feels when it's windy. It is based on a model of how fast a human face loses heat. It was developed through clinical trials in which volunteers were exposed to a variety of temperatures and wind speeds in a refrigerated wind tunnel.

T and v , and we can write $W = f(T, v)$. Table 1 records values of W compiled by the US National Weather Service and the Meteorological Service of Canada.

Table 1 Wind-chill index as a function of air temperature and wind speed

		Wind speed (km/h)										
$T \backslash v$		5	10	15	20	25	30	40	50	60	70	80
	5	4	3	2	1	1	0	-1	-1	-2	-2	-3
Actual temperature (°C)	0	-2	-3	-4	-5	-6	-6	-7	-8	-9	-9	-10
	-5	-7	-9	-11	-12	-12	-13	-14	-15	-16	-16	-17
	-10	-13	-15	-17	-18	-19	-20	-21	-22	-23	-23	-24
	-15	-19	-21	-23	-24	-25	-26	-27	-29	-30	-30	-31
	-20	-24	-27	-29	-30	-32	-33	-34	-35	-36	-37	-38
	-25	-30	-33	-35	-37	-38	-39	-41	-42	-43	-44	-45
	-30	-36	-39	-41	-43	-44	-46	-48	-49	-50	-51	-52
	-35	-41	-45	-48	-49	-51	-52	-54	-56	-57	-58	-60
	-40	-47	-51	-54	-56	-57	-59	-61	-63	-64	-65	-67

For instance, the table shows that if the actual temperature is -5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about -15°C with no wind. So

$f(-5, 50) = -15$

Table 2

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

EXAMPLE 4 In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While many other factors affect economic performance, this model proved to be remarkably accurate. The function Cobb and Douglas used to model production was of the form

$P(L, K) = bL^{\alpha}K^{1-\alpha}$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings). In the Discovery Project following Section 14.3 we will show how the form of Equation 1 follows from certain economic assumptions.

Cobb and Douglas used economic data published by the government to obtain Table 2. They took the year 1899 as a baseline and P , L , and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 values.

Cobb and Douglas used the method of least squares to fit the data of Table 2 to the function

$P(L, K) = 1.01L^{0.75}K^{0.25}$

(See Exercise 81 for the details.)

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.

The production function (1) has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K) \mid L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative.

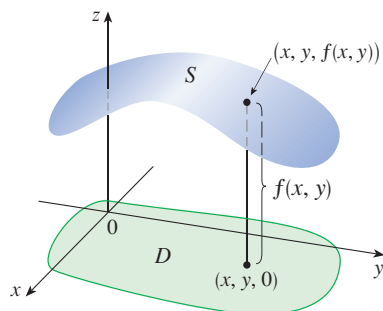


FIGURE 5

Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D , then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D .

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy -plane (see Figure 5).

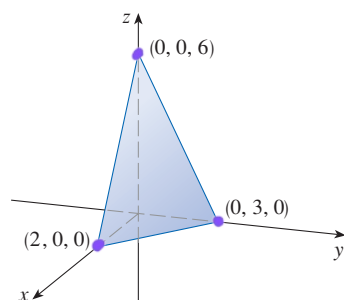


FIGURE 6

EXAMPLE 5 Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

SOLUTION The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x -intercept. Similarly, the y -intercept is 3 and the z -intercept is 6. This helps us sketch the portion of the graph that lies in the first octant in Figure 6.

The function in Example 5 is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of such a function has the equation

$$z = ax + by + c \quad \text{or} \quad ax + by - z + c = 0$$

so it is a plane (see Section 12.5). In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

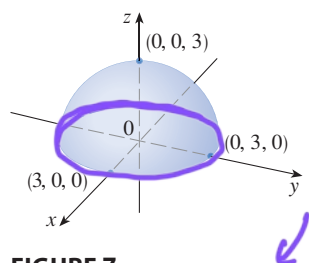


FIGURE 7

Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

EXAMPLE 6 Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION In Example 2 we found that the domain of g is the disk with center $(0, 0)$ and radius 3. The graph of g has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 7).

Ex ⑤ $\underline{f(x,y)} = 6 - 3x - 2y$

$$z = 6 - 3x - 2y$$

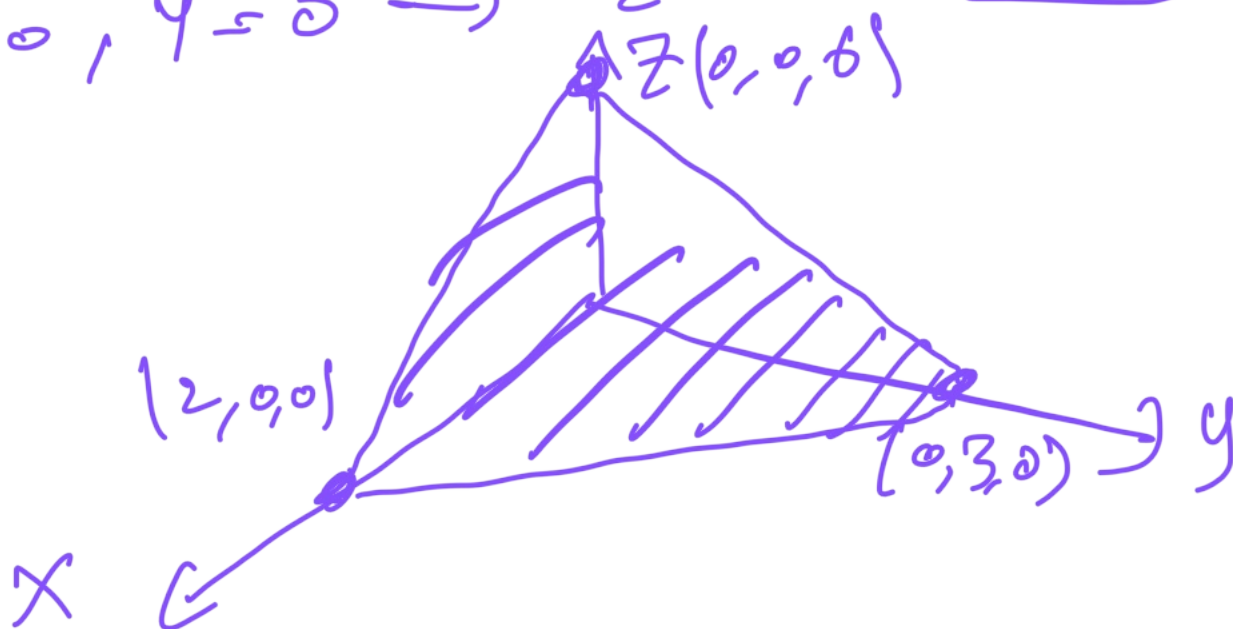
$$\Rightarrow \boxed{3x' + 2y' + z' = 6}$$

↳ linear Equation

$$z=0, y=0 \rightarrow 3x=6 \rightarrow \boxed{x=2}$$

$$x=0, z=0 \rightarrow 2y=6 \rightarrow \boxed{y=3}$$

$$x=0, y=0 \rightarrow z=6 \rightarrow \boxed{z=6}$$



NOTE An entire sphere can't be represented by a single function of x and y . As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

EXAMPLE 7 Use a computer to draw the graph of the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

SOLUTION Figure 8 shows the graph of P for values of the labor L and capital K that lie between 0 and 300. The computer has drawn the surface by plotting vertical traces. We see from these traces that the value of the production P increases as either L or K increases, as expected.

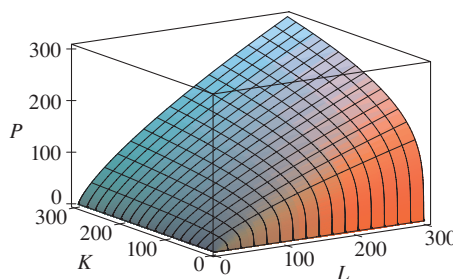


FIGURE 8

EXAMPLE 8 Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

SOLUTION Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y) , so the domain is $\mathbb{R}^2$, the entire xy -plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geq 0$ and $y^2 \geq 0$, so $h(x, y) \geq 0$ for all x and y .] The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid that we sketched in Example 12.6.4. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 9).

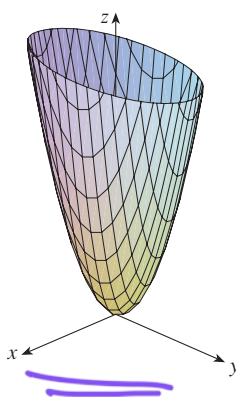


FIGURE 9

Graph of $h(x, y) = 4x^2 + y^2$

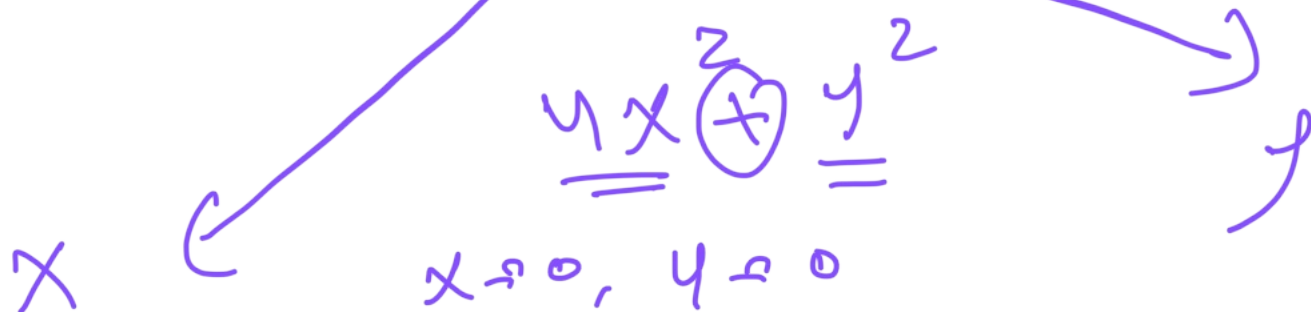
Many software applications are available for graphing functions of two variables. In some programs, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k .

Ex 8

$$h(x, y) = 4x^2 + y^2$$

$$z = 4x^2 + y^2$$

↳ elliptic paraboloid



Domain $\Rightarrow \mathbb{R}^2 \quad (x, y)$
Range $\Rightarrow [0, \infty)$

Figure 10 shows computer-generated graphs of several functions. Notice that we get an especially good picture of a function when rotation is used to give views from different vantage points. In parts (a) and (b) the graph of f is very flat and close to the xy -plane except near the origin; this is because $e^{-x^2-y^2}$ is very small when x or y is large.

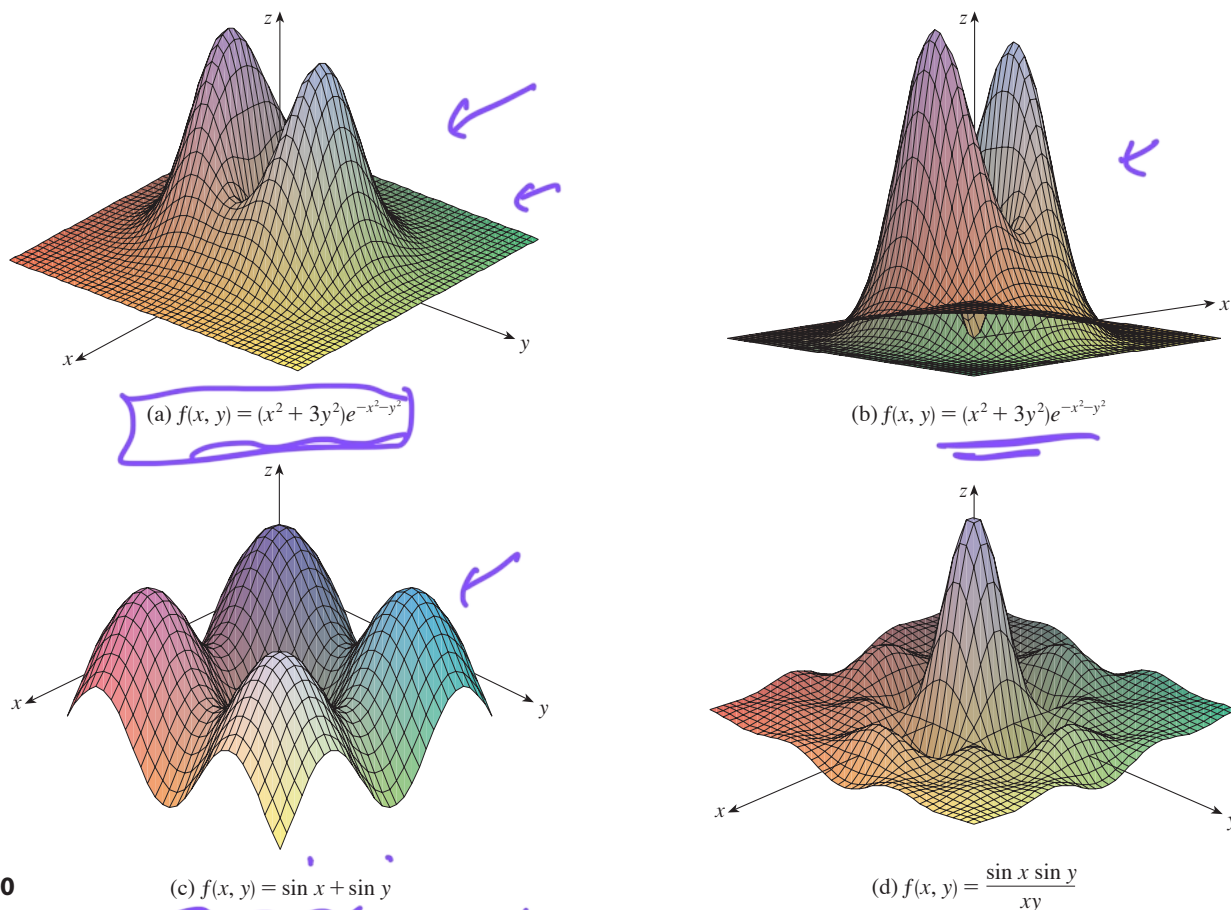


FIGURE 10

Level Curves and Contour Maps

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a *contour map* on which points of constant elevation are joined to form *contour curves*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k . In other words, it is a curve in the xy -plane that shows where the graph of f has height k (above or below the xy -plane). A collection of level curves is called a **contour map**. Contour maps are most descriptive when the level curves

$f(x, y) = k$ are drawn for equally spaced values of k , and we assume that this is the case unless indicated otherwise.

You can see from Figure 11 the relation between level curves and horizontal traces. The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy -plane. So if you draw a contour map of a function and visualize the level curves being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph. The surface is steeper where the level curves are close together and somewhat flatter where they are farther apart.

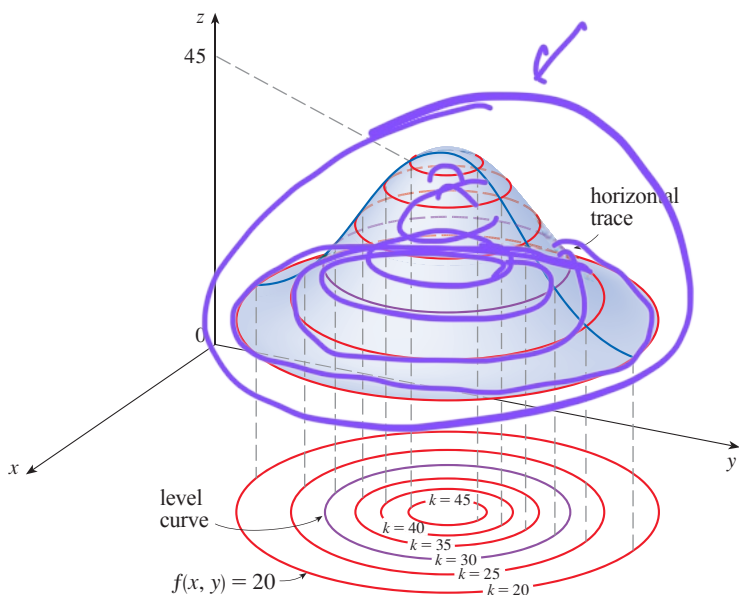


FIGURE 11

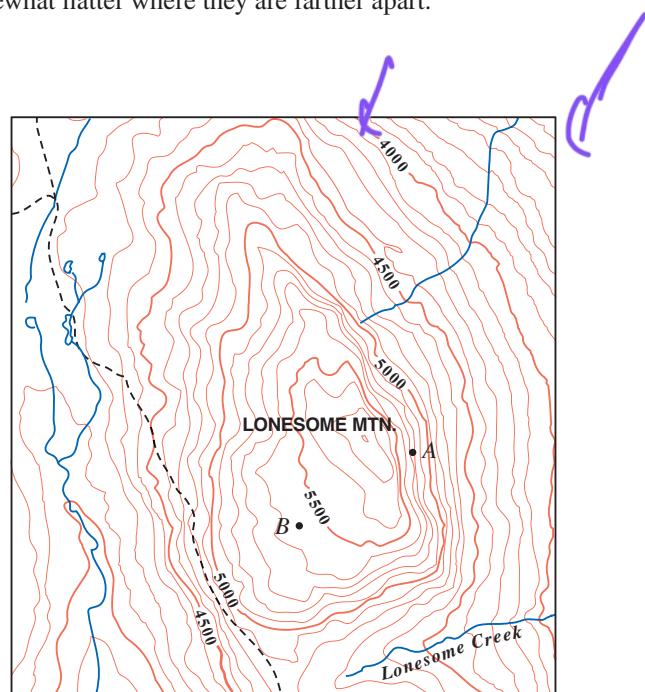


FIGURE 12

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12. The level curves are curves of constant elevation above sea level. If you walk along one of these contour lines, you neither ascend nor descend. Another common example is the temperature function introduced in the opening paragraph of this section. Here the level curves are called **isotherms**; they join locations with the same temperature. Figure 13 shows a weather map of the world indicating the average July temperatures. The isotherms are the curves that separate the colored bands.

In weather maps of atmospheric pressure at a given time as a function of longitude and latitude, the level curves are called **isobars**; they join locations with the same pressure (see Exercise 34). Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure and are strongest where the isobars are tightly packed.

A contour map of worldwide precipitation is shown in Figure 14. Here the level curves are not labeled but they separate the colored regions and the amount of precipitation in each region is indicated in the color key.

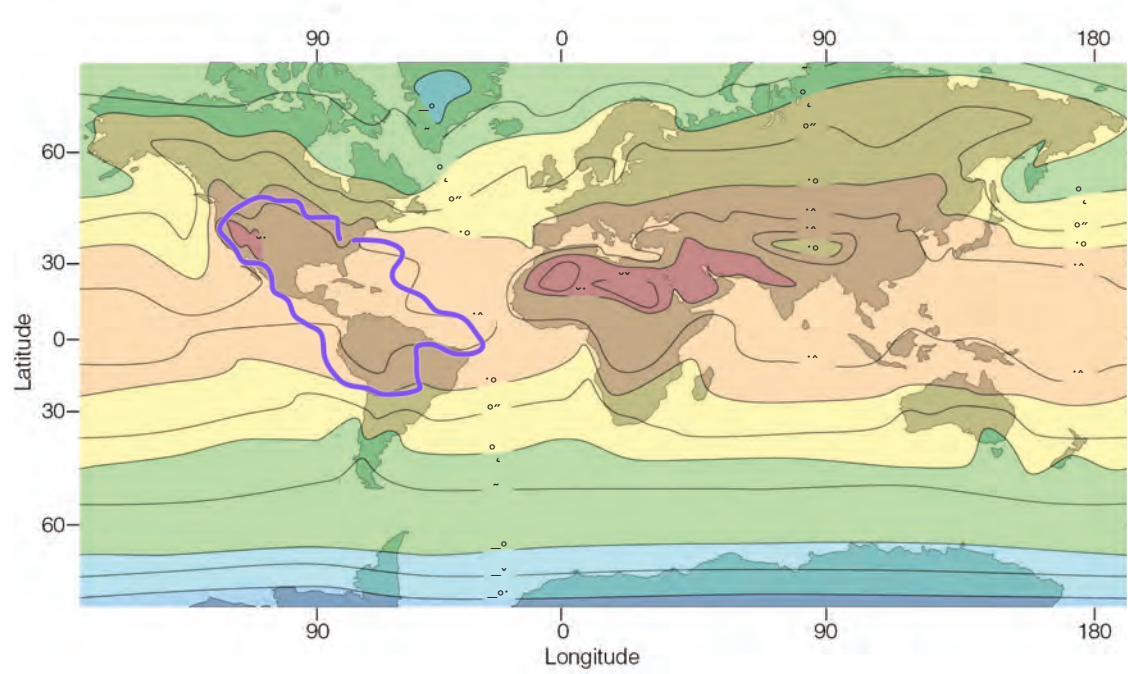


FIGURE 13 Average air temperature near sea level in July (degrees Celsius)

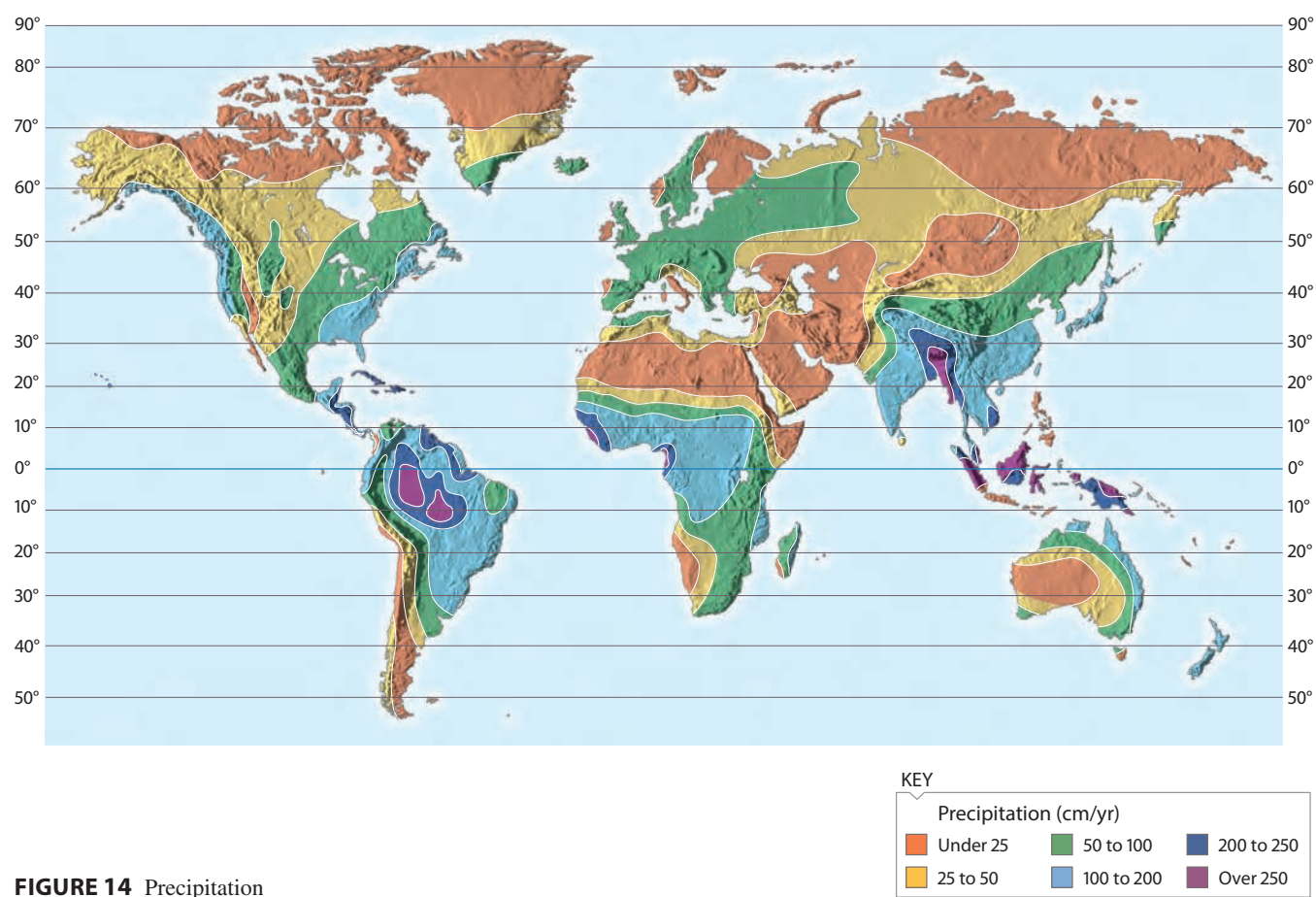


FIGURE 14 Precipitation

EXAMPLE 9 A contour map for a function f is shown in Figure 15. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

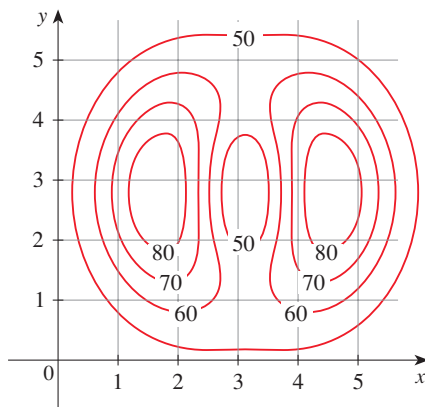


FIGURE 15

SOLUTION The point $(1, 3)$ lies partway between the level curves with z -values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that

$$f(4, 5) \approx 56$$

EXAMPLE 10 Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

SOLUTION The level curves are

$$6 - 3x - 2y = k \quad \text{or} \quad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$, and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 16. For equally spaced values of k the level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6).

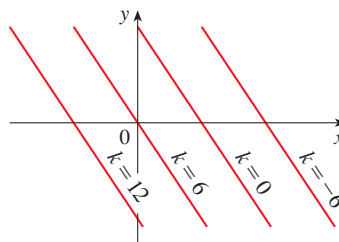


FIGURE 16
Contour map of
 $f(x, y) = 6 - 3x - 2y$

EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \quad \text{for } k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \quad \text{or} \quad x^2 + y^2 = 9 - k^2$$

Ex (11) ✓

$$g(x, y) = \sqrt{a - x^2 - y^2}$$

$$k = 0, 1, 2, 3$$

$$k \neq \sqrt{a - x^2 - y^2}$$

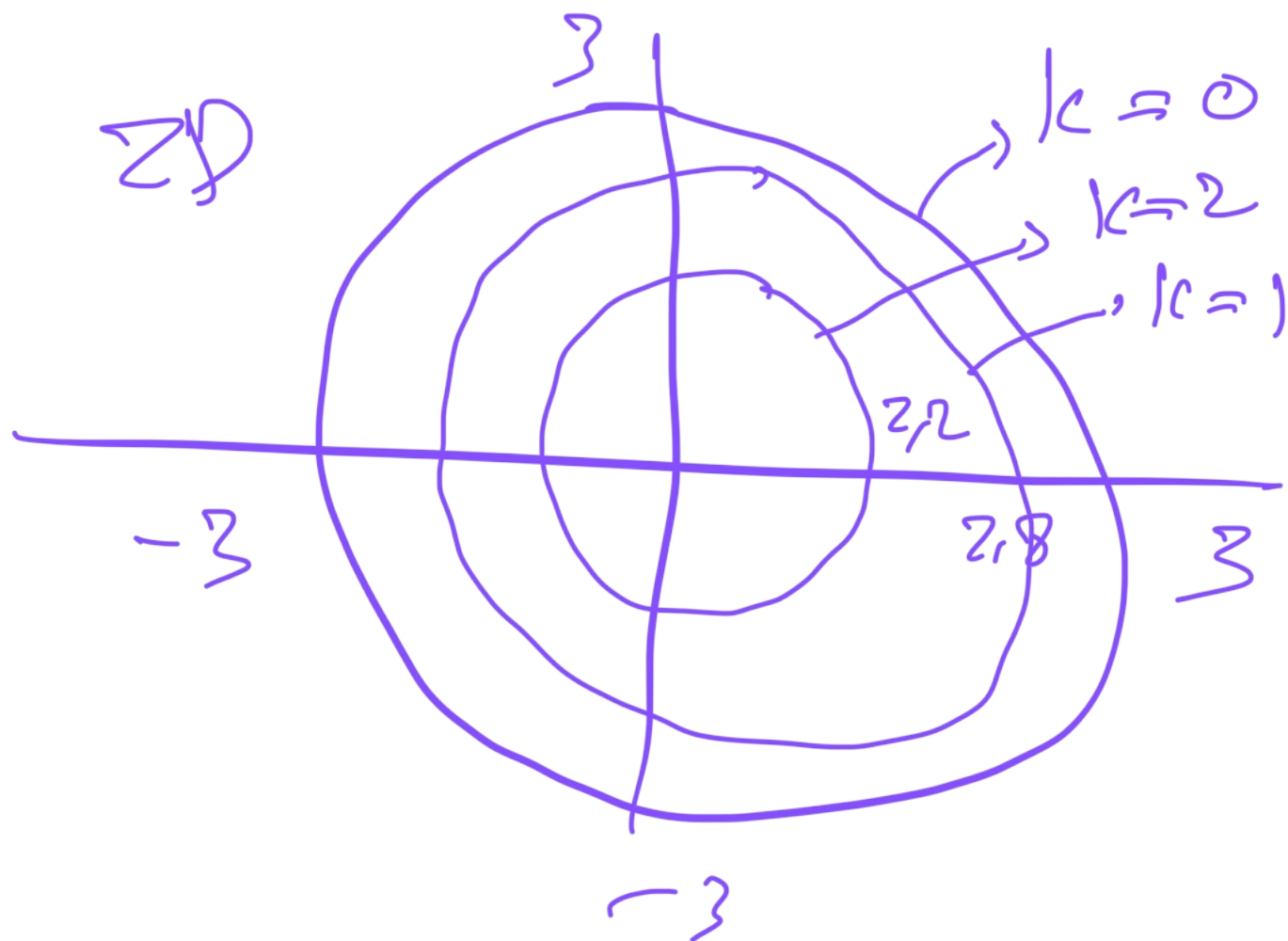
$$(k)^2 = \left(\sqrt{a - x^2 - y^2} \right)^2$$

$$\Rightarrow k^2 = a - \underline{x^2 - y^2}$$

$$\boxed{x^2 + y^2 = a - k^2}$$

Circle
Eq

$$\begin{aligned} k=0 &\rightarrow x^2 + y^2 = a \rightarrow r=3 \\ k=1 &\rightarrow x^2 + y^2 = 8 \rightarrow r=\sqrt{8} = 2.8 \\ k=2 &\rightarrow x^2 + y^2 = 5 \rightarrow r=\sqrt{5} = \underline{2.2} \end{aligned}$$



$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

Equation of sphere

$$z = \sqrt{9 - x^2 - y^2}$$



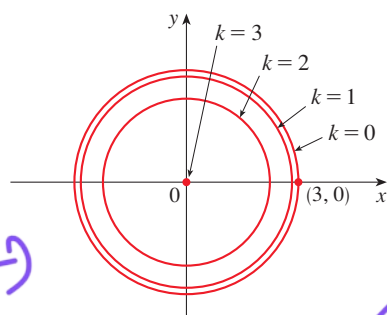


FIGURE 17
Contour map of
 $g(x, y) = \sqrt{9 - x^2 - y^2}$

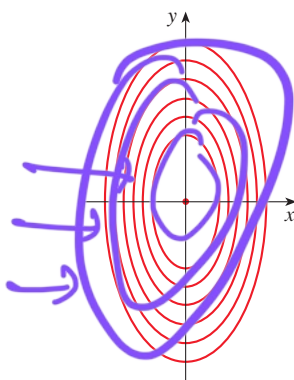
This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown in Figure 17. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7.

EXAMPLE 12 Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

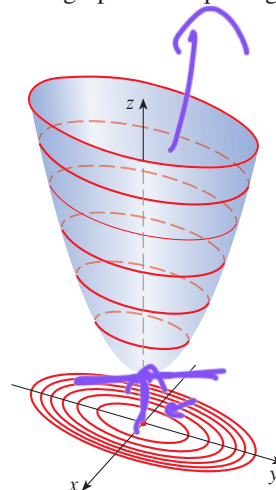
SOLUTION The level curves are

$$4x^2 + y^2 + 1 = k \quad \text{or} \quad \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{k-1} = 1$$

which, for $k > 1$, describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$. Figure 18(a) shows a contour map of h drawn by a computer. Figure 18(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure 18 how the graph of h is put together from the level curves.



(a) Contour map



(b) Horizontal traces are raised level curves.

FIGURE 18

The graph of $h(x, y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.

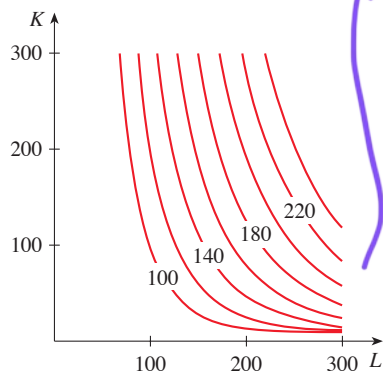


FIGURE 19

EXAMPLE 13 Plot level curves for the Cobb-Douglas production function of Example 4.

SOLUTION In Figure 19 we use a computer to draw a contour plot for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

Level curves are labeled with the value of the production P . For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of $P = 140$. We see that, for a fixed value of P , as L increases K decreases, and vice versa.

For some purposes, a contour map is more useful than a graph. That is certainly true in Example 13. (Compare Figure 19 with Figure 8.) It is also true in estimating function values, as in Example 9.

Ex 12

$$h(x, y) = 4x^2 + y^2 + 1$$

$$4x^2 + y^2 + 1 = k$$

$$\frac{4x^2}{4} + \frac{y^2}{1} = \frac{k-1}{4}$$

$$\boxed{\frac{x^2}{1} + \frac{y^2}{4} = \frac{k-1}{4}}$$

elliptic

circle

$$x^2 + y^2 = r^2 \rightarrow \text{circle}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \text{elliptic}$$

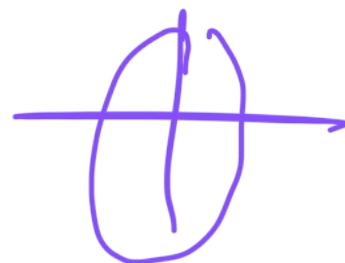
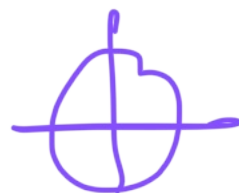


Figure 20 shows some computer-generated level curves together with the corresponding computer-generated graphs. Notice that the level curves in part (c) crowd together near the origin. That corresponds to the fact that the graph in part (d) is very steep near the origin.

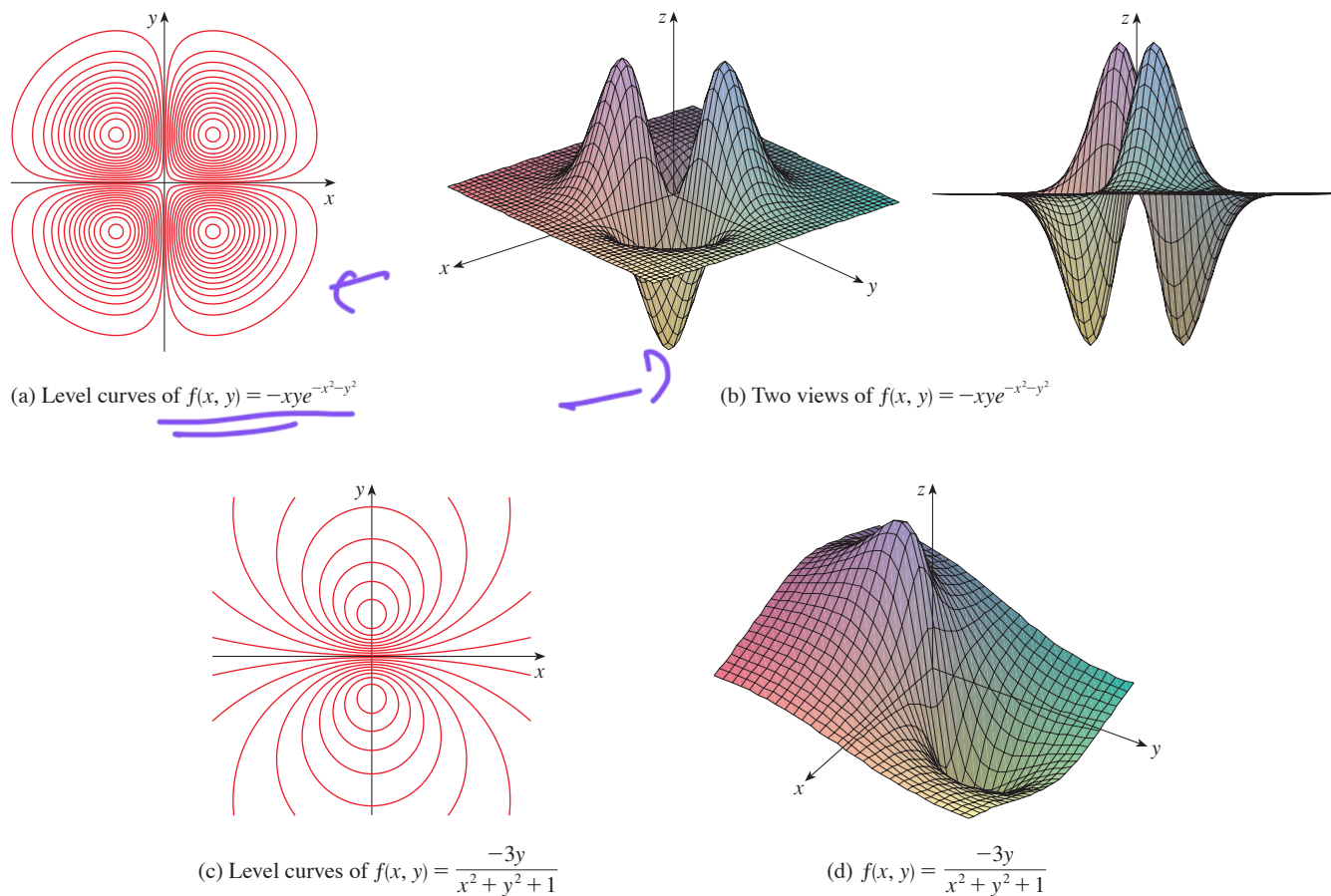


FIGURE 20

Functions of Three or More Variables

A **function of three variables**, f , is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t , so we could write $T = f(x, y, t)$.

EXAMPLE 14 Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

SOLUTION The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$.

Ex 14

$$f(x, y, z) = \underline{\ln(z-y)} + \underbrace{xy \sin z}$$

$$z-y > 0 \Rightarrow \boxed{z > y}$$

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid \boxed{z > y}\}$$

$$\textcircled{3D} \rightarrow z = y \quad \text{plane}$$

$$2D \quad x = y \rightarrow \text{line}$$

$$3D \quad x = y \quad \boxed{\phantom{\text{line}}}$$

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure 21.) Thus, as (x, y, z) varies over any sphere with center O , the value of $f(x, y, z)$ remains fixed.

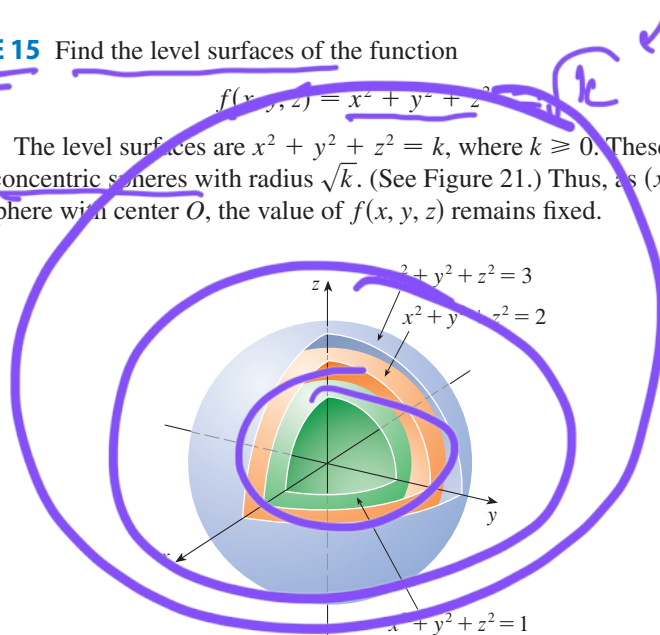


FIGURE 21

EXAMPLE 16 Describe the level surfaces of the function

$$f(x, y, z) = x^2 - y - z^2$$

SOLUTION The level surfaces are $x^2 - y - z^2 = k$, or $y = x^2 - z^2 - k$, a family of hyperbolic paraboloids. Figure 22 shows the level surfaces for $k = 0$ and $k = \pm 5$.

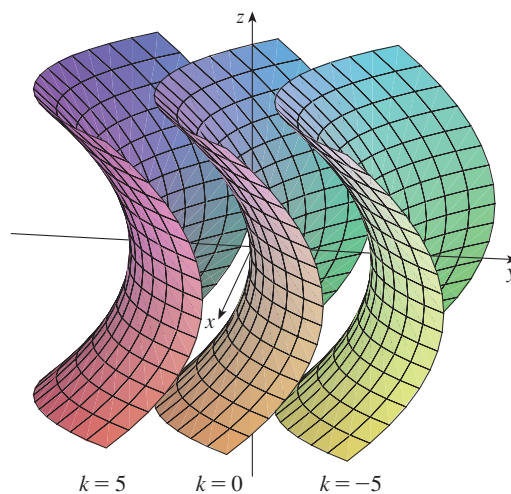


FIGURE 22

Functions of any number of variables can be considered. A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \dots, x_n)$ to an n -tuple $(x_1, x_2, \dots, x_n)$.

of real numbers. We denote by $\mathbb{R}^n$ the set of all such n -tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the i th ingredient, and x_i units of the i th ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \dots, x_n$:

$$\boxed{3} \quad C = f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \dots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \dots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n .

In view of the one-to-one correspondence between points $(x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ in V_n , we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \dots, x_n$
2. As a function of a single point variable $(x_1, x_2, \dots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$

We will see that all three points of view are useful.

14.1 Exercises

1. If $f(x, y) = x^2y/(2x - y^2)$, find
 - (a) $f(1, 3)$
 - (b) $f(-2, -1)$
 - (c) $f(x + h, y)$
 - (d) $f(x, x)$
2. If $g(x, y) = x \sin y + y \sin x$, find
 - (a) $g(\pi, 0)$
 - (b) $g(\pi/2, \pi/4)$
 - (c) $g(0, y)$
 - (d) $g(x, y + h)$
3. Let $g(x, y) = x^2 \ln(x + y)$.
 - (a) Evaluate $g(3, 1)$.
 - (b) Find and sketch the domain of g .
 - (c) Find the range of g .
4. Let $h(x, y) = e^{\sqrt{y-x^2}}$.
 - (a) Evaluate $h(-2, 5)$.
 - (b) Find and sketch the domain of h .
 - (c) Find the range of h .
5. Let $F(x, y, z) = \sqrt{y} - \sqrt{x^2 - 2z}$.
 - (a) Evaluate $F(3, 4, 1)$.
 - (b) Find and describe the domain of F .
6. Let $f(x, y, z) = \ln(z - \sqrt{x^2 + y^2})$.
 - (a) Evaluate $f(4, -3, 6)$.
 - (b) Find and describe the domain of f .

7–16 Find and sketch the domain of the function.

7. $f(x, y) = \sqrt{x-2} + \sqrt{y-1}$
8. $f(x, y) = \sqrt[4]{x-3y}$

$$9. q(x, y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$$

$$10. g(x, y) = \ln(x^2 + y^2 - 9)$$

$$11. g(x, y) = \frac{x-y}{x+y}$$

$$12. g(x, y) = \frac{\ln(2-x)}{1-x^2-y^2}$$

$$13. p(x, y) = \frac{\sqrt{xy}}{x+1}$$

$$14. f(x, y) = \sin^{-1}(x+y)$$

$$15. f(x, y, z) = \sqrt{4-x^2} + \sqrt{9-y^2} + \sqrt{1-z^2}$$

$$16. f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$$

17. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.0072w^{0.425}h^{0.725}$$

where w is the weight (in kilograms), h is the height (in centimeters), and S is measured in square meters.

- (a) Find $f(73, 178)$ and interpret it.
- (b) What is your own surface area?

18. A manufacturer has modeled its yearly production function P (the monetary value of its entire production in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Find $P(120, 20)$ and interpret it.

19. In Example 3 we considered the function $W = f(T, v)$, where W is the wind-chill index, T is the actual temperature, and v is the wind speed. A numerical representation is given in Table 1.

- What is the value of $f(-15, 40)$? What is its meaning?
- Describe in words the meaning of the question “For what value of v is $f(-20, v) = -30$?” Then answer the question.
- Describe in words the meaning of the question “For what value of T is $f(T, 20) = -49$?” Then answer the question.
- What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.
- What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.

20. The *temperature-humidity index* I (or *humidex*, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h , so we can write $I = f(T, h)$. The following table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

Table 3 Apparent temperature as a function of temperature and humidity

		Relative humidity (%)					
Actual temperature (°C)	h	20	30	40	50	60	70
	20	20	20	20	21	22	23
	25	25	25	26	28	30	32
	30	30	31	34	36	38	41
	35	36	39	42	45	48	51
	40	43	47	51	55	59	63

- What is the value of $f(95, 70)$? What is its meaning?
- For what value of h is $f(90, h) = 100$?
- For what value of T is $f(T, 50) = 88$?
- What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of h .

21. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.

- What is the value of $f(40, 15)$? What is its meaning?
- What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.
- What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

Table 4 Wave height as a function of wind speed and duration

		Duration (hours)						
Wind speed (km/h)	t	5	10	15	20	30	40	50
	20	0.6	0.6	0.6	0.6	0.6	0.6	0.6
	30	1.2	1.3	1.5	1.5	1.5	1.6	1.6
	40	1.5	2.2	2.4	2.5	2.7	2.8	2.8
	60	2.8	4.0	4.9	5.2	5.5	5.8	5.9
	80	4.3	6.4	7.7	8.6	9.5	10.1	10.2
	100	5.8	8.9	11.0	12.2	13.8	14.7	15.3
	120	7.4	11.3	14.4	16.6	19.0	20.5	21.1

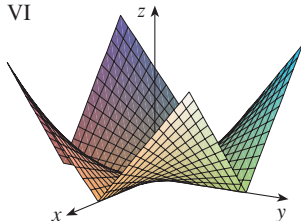
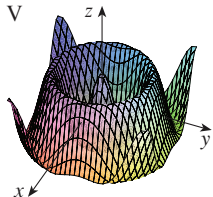
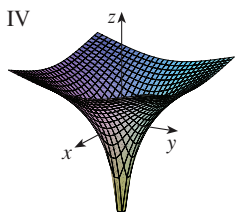
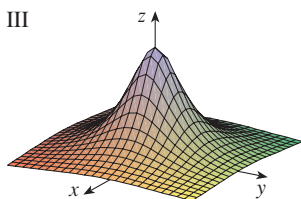
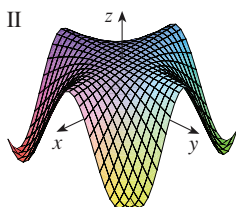
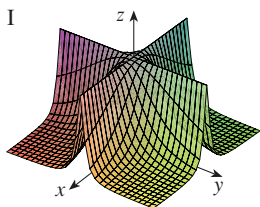
22. A company makes three sizes of cardboard boxes: small, medium, and large. It costs \$2.50 to make a small box, \$4.00 for a medium box, and \$4.50 for a large box. Fixed costs are \$8000.
- Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.
 - Find $f(3000, 5000, 4000)$ and interpret it.
 - What is the domain of f ?

23–31 Sketch the graph of the function.

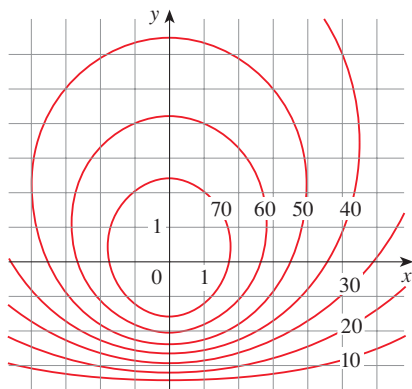
- $f(x, y) = y$
- $f(x, y) = x^2$
- $f(x, y) = 10 - 4x - 5y$
- $f(x, y) = \cos y$
- $f(x, y) = \sin x$
- $f(x, y) = 2 - x^2 - y^2$
- $f(x, y) = x^2 + 4y^2 + 1$
- $f(x, y) = \sqrt{4x^2 + y^2}$
- $f(x, y) = \sqrt{4 - 4x^2 - y^2}$

32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = \frac{1}{1 + x^2 + y^2}$ (b) $f(x, y) = \frac{1}{1 + x^2 y^2}$
 (c) $f(x, y) = \ln(x^2 + y^2)$ (d) $f(x, y) = \cos \sqrt{x^2 + y^2}$
 (e) $f(x, y) = |xy|$ (f) $f(x, y) = \cos(xy)$

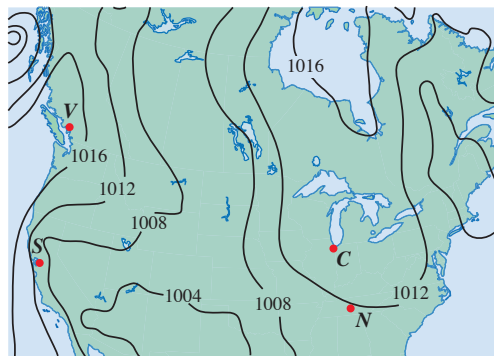


33. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?

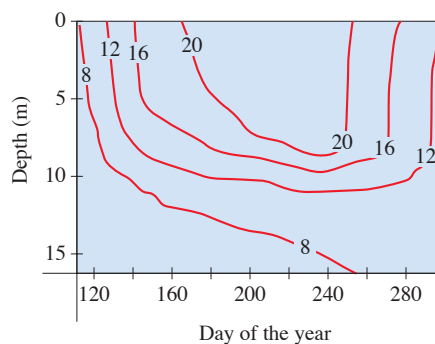


34. Shown is a contour map of atmospheric pressure in North America on a particular day. On the level curves (isobars) the pressure is indicated in millibars (mb).

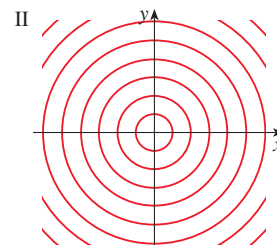
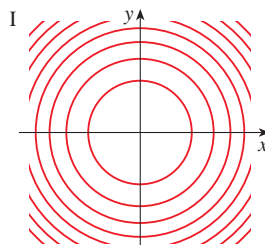
- (a) Estimate the pressure at C (Chicago), N (Nashville), S (San Francisco), and V (Vancouver).
 (b) At which of these locations were the winds strongest? (See the discussion preceding Example 9.)



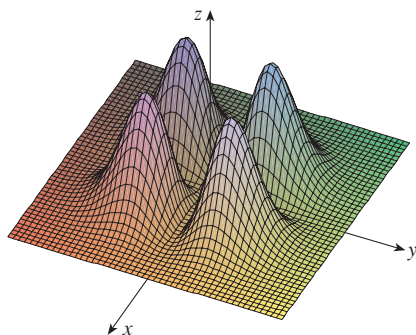
35. Level curves (isotherms) are shown for the typical water temperature (in $^{\circ}\text{C}$) in Long Lake (Minnesota) as a function of depth and time of year. Estimate the temperature in the lake on June 9 (day 160) at a depth of 10 m and on June 29 (day 180) at a depth of 5 m.



36. Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?



37. Locate the points A and B on the map of Lonesome Mountain (Figure 12). How would you describe the terrain near A ? Near B ?
38. Make a rough sketch of a contour map for the function whose graph is shown.



39. The *body mass index* (BMI) of a person is defined by

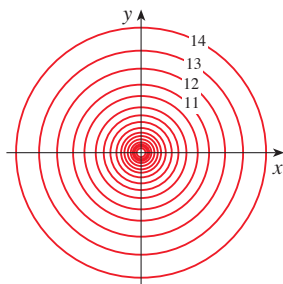
$$B(m, h) = \frac{m}{h^2}$$

where m is the person's mass (in kilograms) and h is the person's height (in meters). Draw the level curves $B(m, h) = 18.5$, $B(m, h) = 25$, $B(m, h) = 30$, and $B(m, h) = 40$. A rough guideline is that a person is underweight if the BMI is less than 18.5; optimal if the BMI lies between 18.5 and 25; overweight if the BMI lies between 25 and 30; and obese if the BMI exceeds 30. Shade the region corresponding to optimal BMI. Does someone who weighs 62 kg and is 152 cm tall fall into the optimal category?

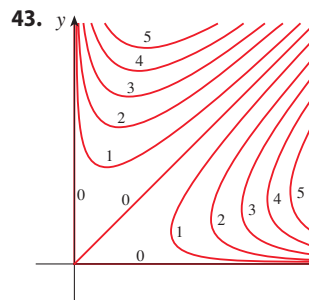
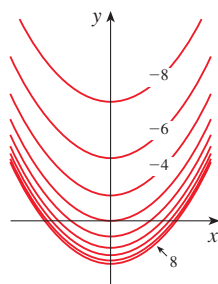
40. The body mass index is defined in Exercise 39. Draw the level curve of this function corresponding to someone who is 200 cm tall and weighs 80 kg. Find the weights and heights of two other people with that same level curve.

- 41–44 A contour map of a function is shown. Use it to make a rough sketch of the graph of f .

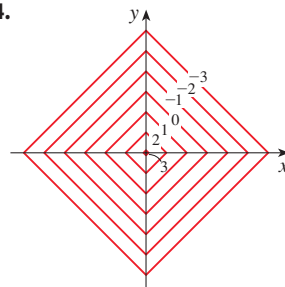
41.



42.



44.



- 45–52 Draw a contour map of the function showing several level curves.

45. $f(x, y) = x^2 - y^2$

46. $f(x, y) = xy$

47. $f(x, y) = \sqrt{x} + y$

48. $f(x, y) = \ln(x^2 + 4y^2)$

49. $f(x, y) = ye^x$

50. $f(x, y) = y - \arctan x$

51. $f(x, y) = \sqrt[3]{x^2 + y^2}$

52. $f(x, y) = y/(x^2 + y^2)$

- 53–54 Sketch both a contour map and a graph of the given function and compare them.

53. $f(x, y) = x^2 + 9y^2$

54. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

55. A thin metal plate, located in the xy -plane, has temperature $T(x, y)$ at the point (x, y) . Sketch some level curves (isothermals) if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}$$

56. If $V(x, y)$ is the electric potential at a point (x, y) in the xy -plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

- 57–60 Graph the function using various domains and viewpoints. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

57. $f(x, y) = xy^2 - x^3$ (monkey saddle)

58. $f(x, y) = xy^3 - yx^3$ (dog saddle)

59. $f(x, y) = e^{-(x^2+y^2)/3}(\sin(x^2) + \cos(y^2))$

60. $f(x, y) = \cos x \cos y$