of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$\boxed{3} \qquad C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

14.1 Exercises

1. If $f(x, y) = x^2y/(2x - y^2)$, find
(a) $f(1, 3)$ (b) $f(-2, -1)$
(c) $f(x + h, y)$ (d) $f(x, x)$

2. If $g(x, y) = x \sin y + y \sin x$, find
(a) $g(\pi, 0)$ (b) $g(\pi/2, \pi/4)$
(c) $g(0, y)$ (d) $g(x, y + h)$

3. Let $g(x, y) = x^2 \ln(x + y)$.
(a) Evaluate $g(3, 1)$.
(b) Find and sketch the domain of g.
(c) Find the range of g.

4. Let $h(x, y) = e^{\sqrt{y - x^2}}$.
(a) Evaluate $h(-2, 5)$.
(b) Find and sketch the domain of h.
(c) Find the range of h.

5. Let $F(x, y, z) = \sqrt{y} - \sqrt{x - 2z}$.
(a) Evaluate $F(3, 4, 1)$.
(b) Find and describe the domain of F.

6. Let $f(x, y, z) = \ln\left(z - \sqrt{x^2 + y^2}\right)$.
(a) Evaluate $f(4, -3, 6)$.
(b) Find and describe the domain of f.

7–16 Find and sketch the domain of the function.

7. $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$

8. $f(x, y) = \sqrt[4]{x - 3y}$

9. $g(x, y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$

10. $g(x, y) = \ln(x^2 + y^2 - 9)$

11. $g(x, y) = \dfrac{x - y}{x + y}$

12. $g(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$

13. $p(x, y) = \dfrac{\sqrt{xy}}{x + 1}$

14. $f(x, y) = \sin^{-1}(x + y)$

15. $f(x, y, z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$

16. $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$

17. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.0072w^{0.425}h^{0.725}$$

where w is the weight (in kilograms), h is the height (in centimeters), and S is measured in square meters.
(a) Find $f(73, 178)$ and interpret it.
(b) What is your own surface area?

Range صفحة جانبية

[1] $\sqrt{\ }$ → range: 0 ✓, + ✓, − ✗

Range $[0, \infty)$

---

[2] ln → range: 0 ✓, + ✓, − ✓

Range $\mathbb{R}$

---

[3] أُس $e^{\square}$ → range: 0 ✓, + ✓, − ✗

$e^{-\infty} = 0$

Range $(0, \infty)$

[3] $g(x,y) = x^2 \ln(x+y)$

a) $g(3,1)$ $\quad x=3, \; y=1$

$g(3,1) = 3^2 \ln(3+1) = \boxed{9 \ln 4}$

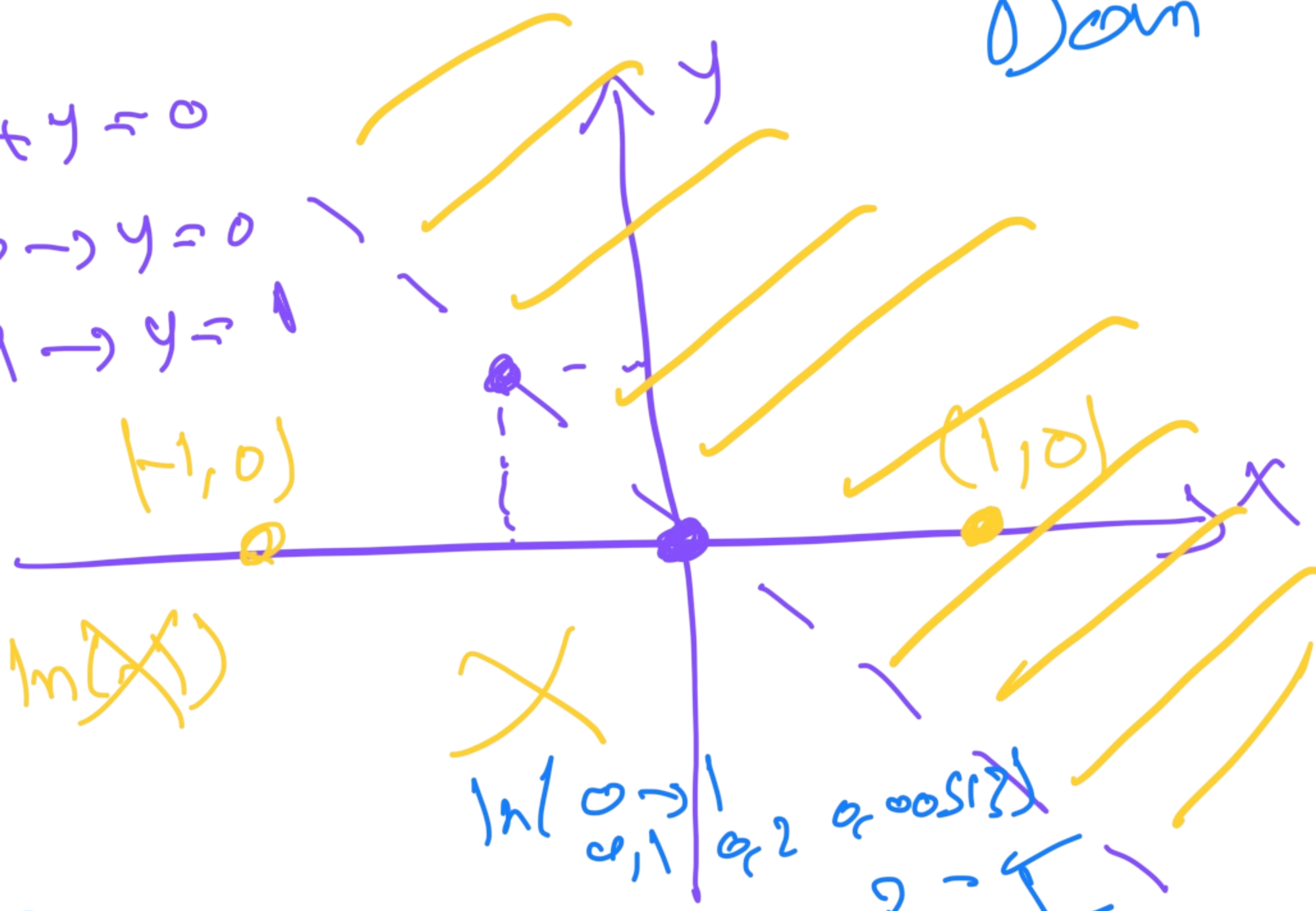
b) $x+y > 0 \longrightarrow \boxed{y > -x}$

Dom

$x+y=0$

$x=0 \rightarrow y=0$

$x=-1 \rightarrow y=1$



c) Range $\underline{\mathbb{R}}$

$\boxed{4}$ $h(x,y) = e^{\sqrt{y-x^2}}$

a) $h(-2,5) \rightarrow e^{\sqrt{5-(-2)^2}}$

x y

$= e^{\sqrt{5-4=1}}$ $\boxed{e}$

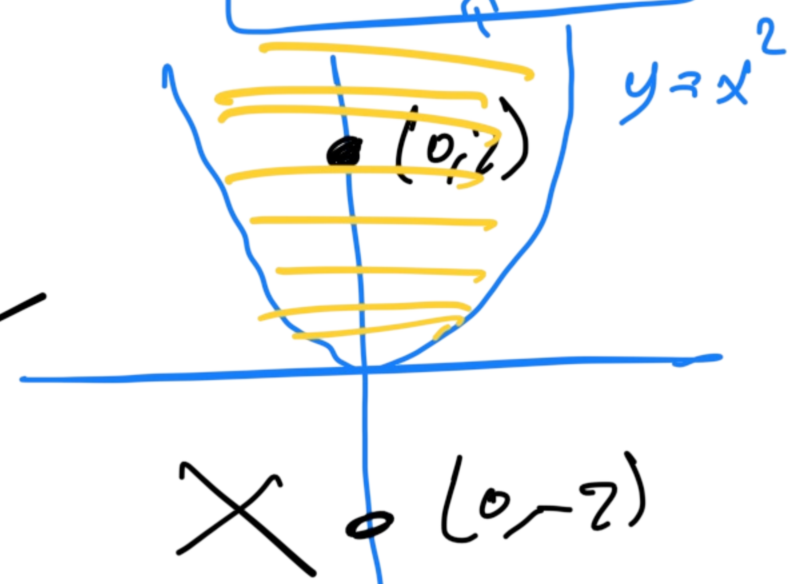
b) Dom $\sqrt{y-x^2}$

$y - x^2 \geq 0 \Rightarrow \boxed{y \geq x^2}$

$y = x^2 \rightarrow$ parabola

$\sqrt{2-0} = \sqrt{2}$ ✓

$\cancel{\sqrt{-2}}$



$h = -\infty$

c) Range $e \to 0$, $+$, $-x$

$e^0 = 1$

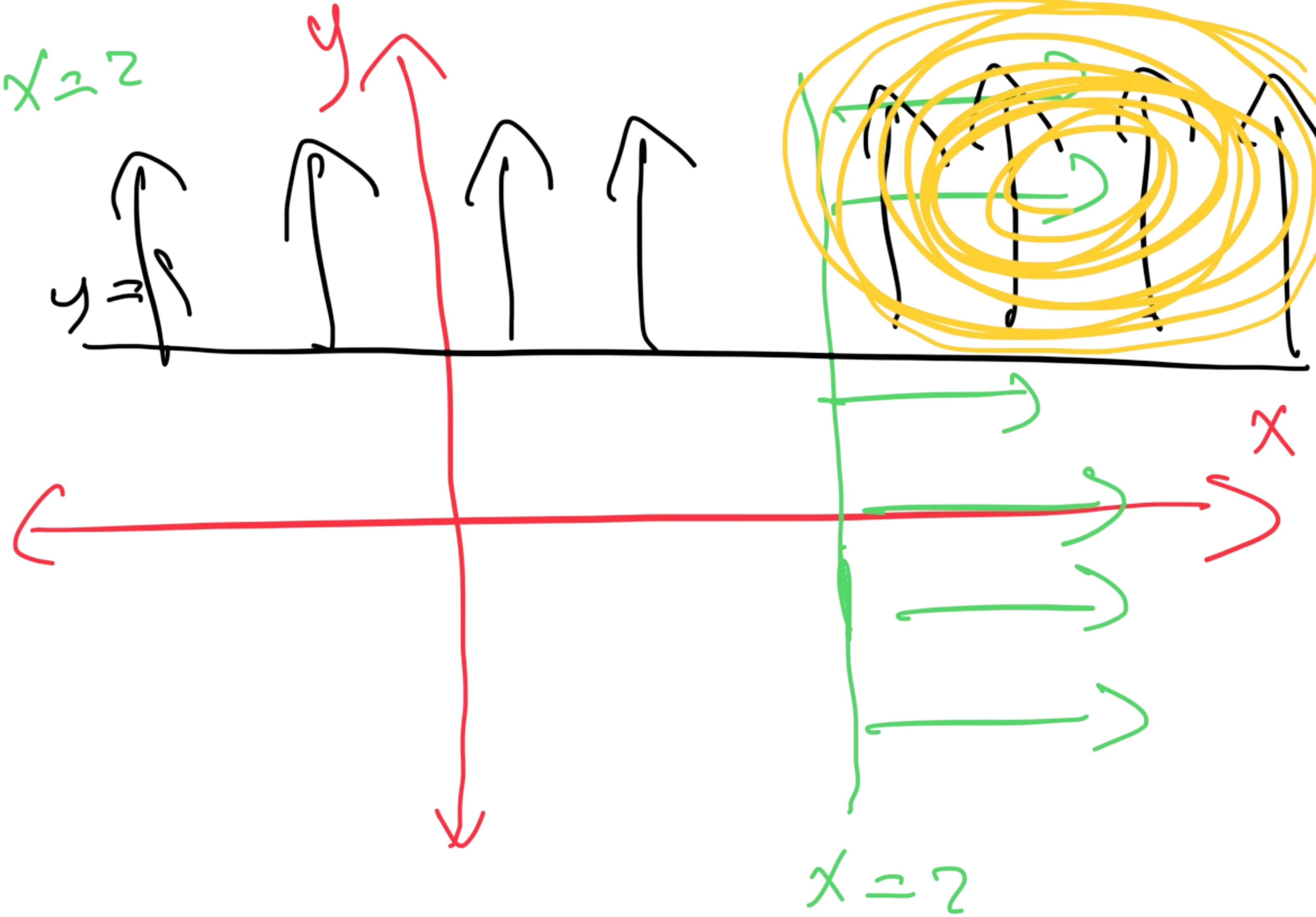
$\boxed{[1,\infty)}$

Q7 $f(x,y) = \sqrt{x-2} + \sqrt{y-1}$

$x-2 \geq 0$ $\quad$ $y-1 \geq 0$

$\boxed{x \geq 2, \; y \geq 1}$ Domain

$y=1$



Q_a $\quad g(x,y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$

$\boxed{x \geq 0}$

$4 - 4x^2 - y^2 \geq 0$

$x = 0$

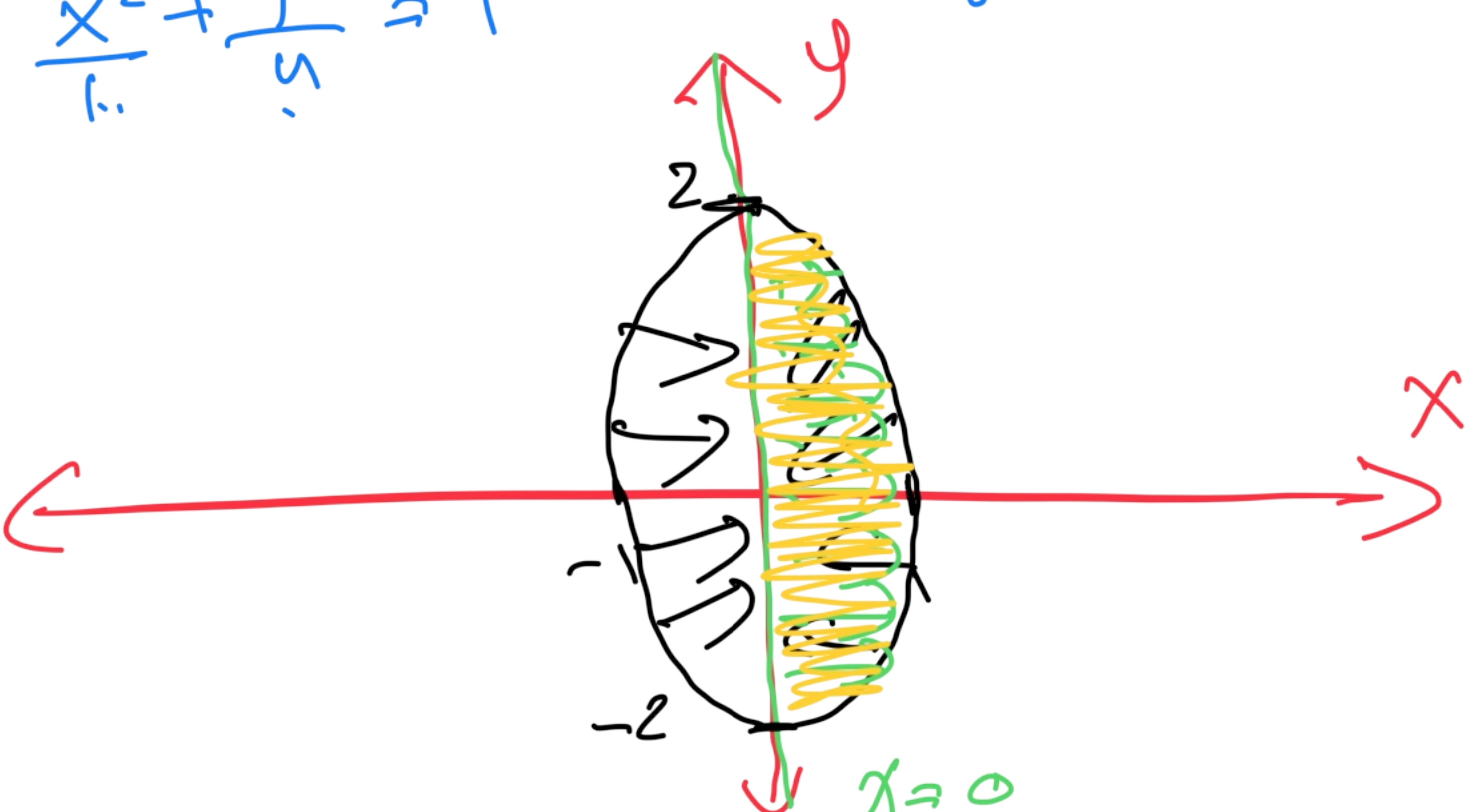
$$\frac{4}{4} \geq \frac{4x^2}{4} + \frac{y^2}{4}$$

$$\boxed{1 \geq \frac{x^2}{1} + \frac{y^2}{4}}$$

$\sqrt{1} = 1$

$\sqrt{4} = 2$

$\frac{x^2}{1} + \frac{y^2}{4} = 1 \longrightarrow$ elliptic shape

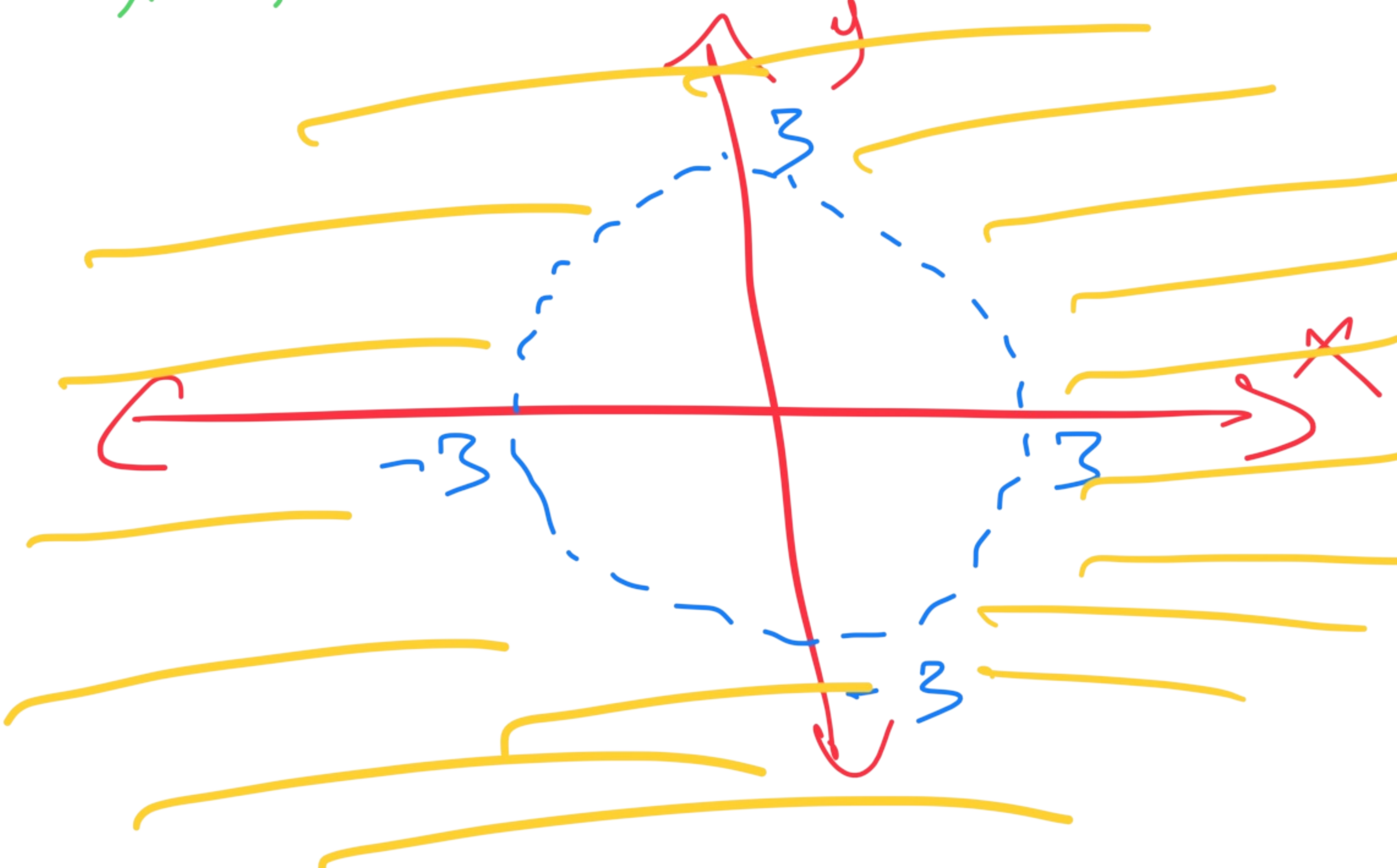


Q10 $g(x,y) = \ln(x^2+y^2-a)$

$x^2+y^2-a > 0$

$x^2+y^2 > a$

$x^2+y^2 = a \longrightarrow$ Eq circle



18. A manufacturer has modeled its yearly production function P (the monetary value of its entire production in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Find $P(120, 20)$ and interpret it.

19. In Example 3 we considered the function $W = f(T, v)$, where W is the wind-chill index, T is the actual temperature, and v is the wind speed. A numerical representation is given in Table 1.

(a) What is the value of $f(-15, 40)$? What is its meaning?
(b) Describe in words the meaning of the question "For what value of v is $f(-20, v) = -30$?" Then answer the question.
(c) Describe in words the meaning of the question "For what value of T is $f(T, 20) = -49$?" Then answer the question.
(d) What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.
(e) What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.

20. The *temperature-humidity index* I (or humidex, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h, so we can write $I = f(T, h)$. The following table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

Table 3 **Apparent temperature as a function of temperature and humidity**

Relative humidity (%); Actual temperature (°C)

T \ h	20	30	40	50	60	70
20	20	20	20	21	22	23
25	25	25	26	28	30	32
30	30	31	34	36	38	41
35	36	39	42	45	48	51
40	43	47	51	55	59	63

(a) What is the value of $f(95, 70)$? What is its meaning?
(b) For what value of h is $f(90, h) = 100$?
(c) For what value of T is $f(T, 50) = 88$?
(d) What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of h.

21. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.

(a) What is the value of $f(40, 15)$? What is its meaning?
(b) What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.
(c) What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

Table 4 **Wave heights as a function of wind speed and duration**

Duration (hours); Wind speed (km/h)

v \ t	5	10	15	20	30	40	50
20	0.6	0.6	0.6	0.6	0.6	0.6	0.6
30	1.2	1.3	1.5	1.5	1.5	1.6	1.6
40	1.5	2.2	2.4	2.5	2.7	2.8	2.8
60	2.8	4.0	4.9	5.2	5.5	5.8	5.9
80	4.3	6.4	7.7	8.6	9.5	10.1	10.2
100	5.8	8.9	11.0	12.2	13.8	14.7	15.3
120	7.4	11.3	14.4	16.6	19.0	20.5	21.1

22. A company makes three sizes of cardboard boxes: small, medium, and large. It costs \$2.50 to make a small box, \$4.00 for a medium box, and \$4.50 for a large box. Fixed costs are \$8000.

(a) Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.
(b) Find $f(3000, 5000, 4000)$ and interpret it.
(c) What is the domain of f?

23–31 Sketch the graph of the function.

23. $f(x, y) = y$

24. $f(x, y) = x^2$

25. $f(x, y) = 10 - 4x - 5y$

26. $f(x, y) = \cos y$

27. $f(x, y) = \sin x$

28. $f(x, y) = 2 - x^2 - y^2$

29. $f(x, y) = x^2 + 4y^2 + 1$

30. $f(x, y) = \sqrt{4x^2 + y^2}$

31. $f(x, y) = \sqrt{4 - 4x^2 - y^2}$

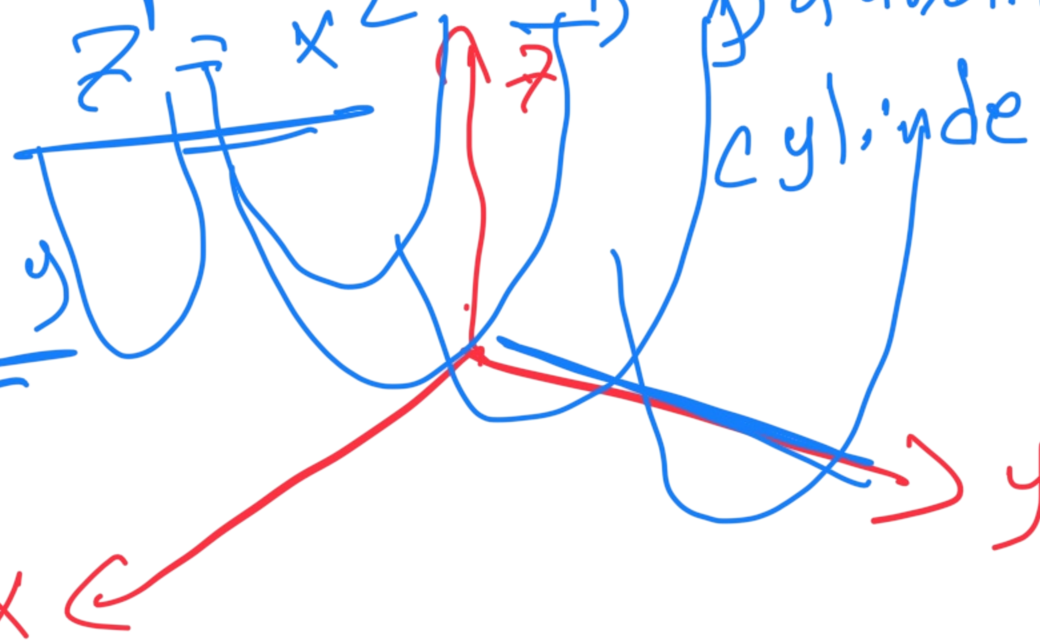
Q24 $f(x,y) = x^2$

$z = x^2$

3D

missing y

parabolic cylinder



Q25 $f(x,y) = 10 - 4x - 5y$

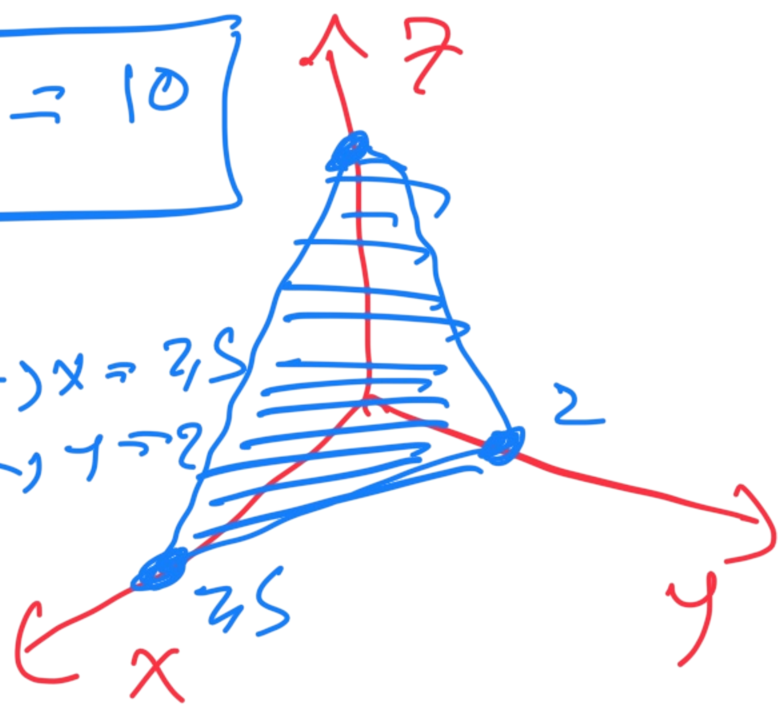
$$\boxed{4x + 5y + z = 10}$$

linear equation

$y=0, z=0 \rightarrow 4x = 10 \rightarrow x = 2.5$

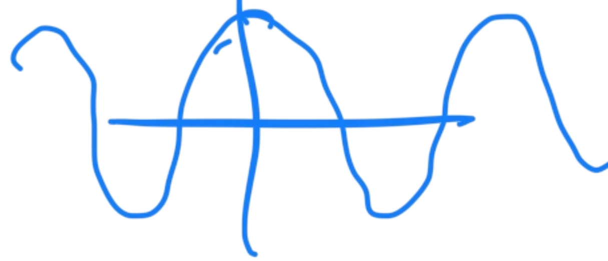
$x=0, z=0 \rightarrow 5y = 10 \rightarrow y = 2$

$x=0, y=0 \rightarrow z = 10$



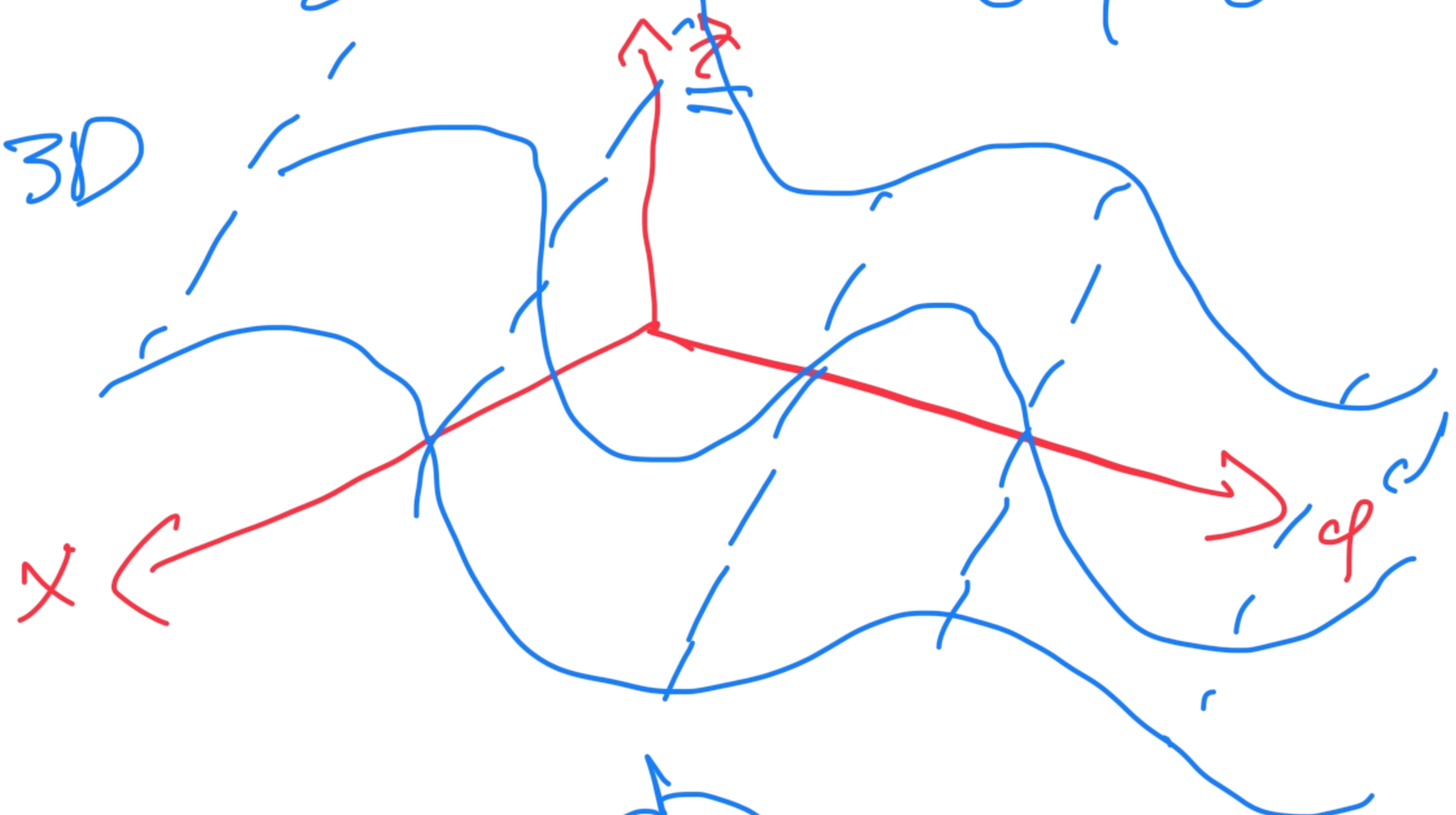
Q 26 $f(x,y) = \cos y$

2D $y = \cos x$

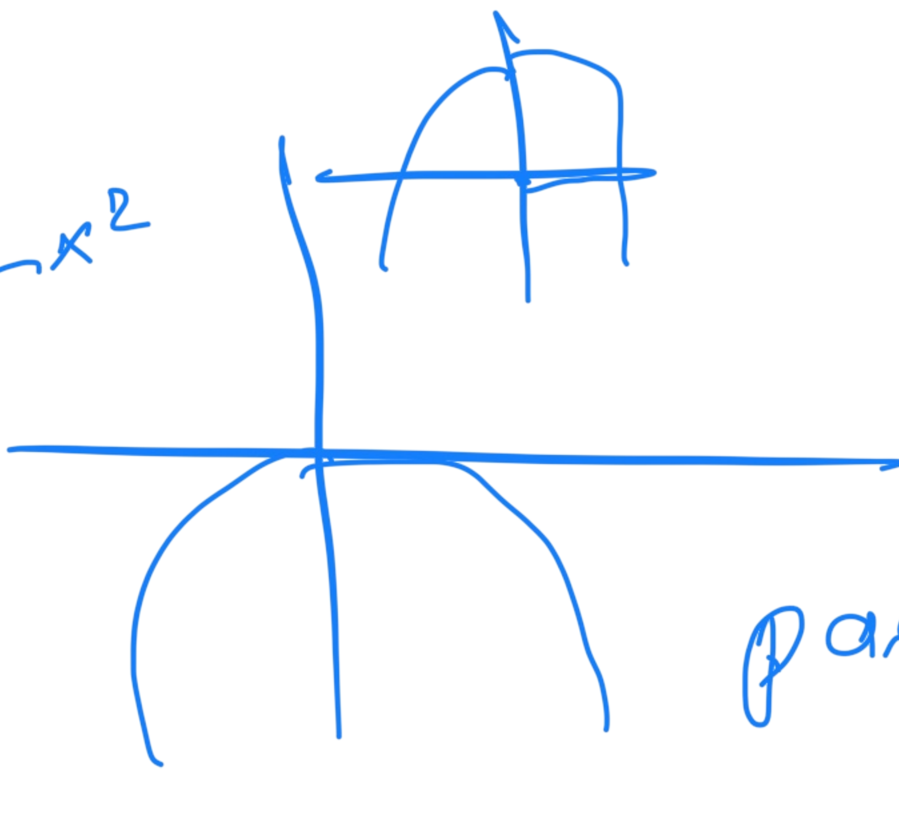


$z = \cos y$

3D



$y = 2 - x^2$



$y = -x^2$

2D

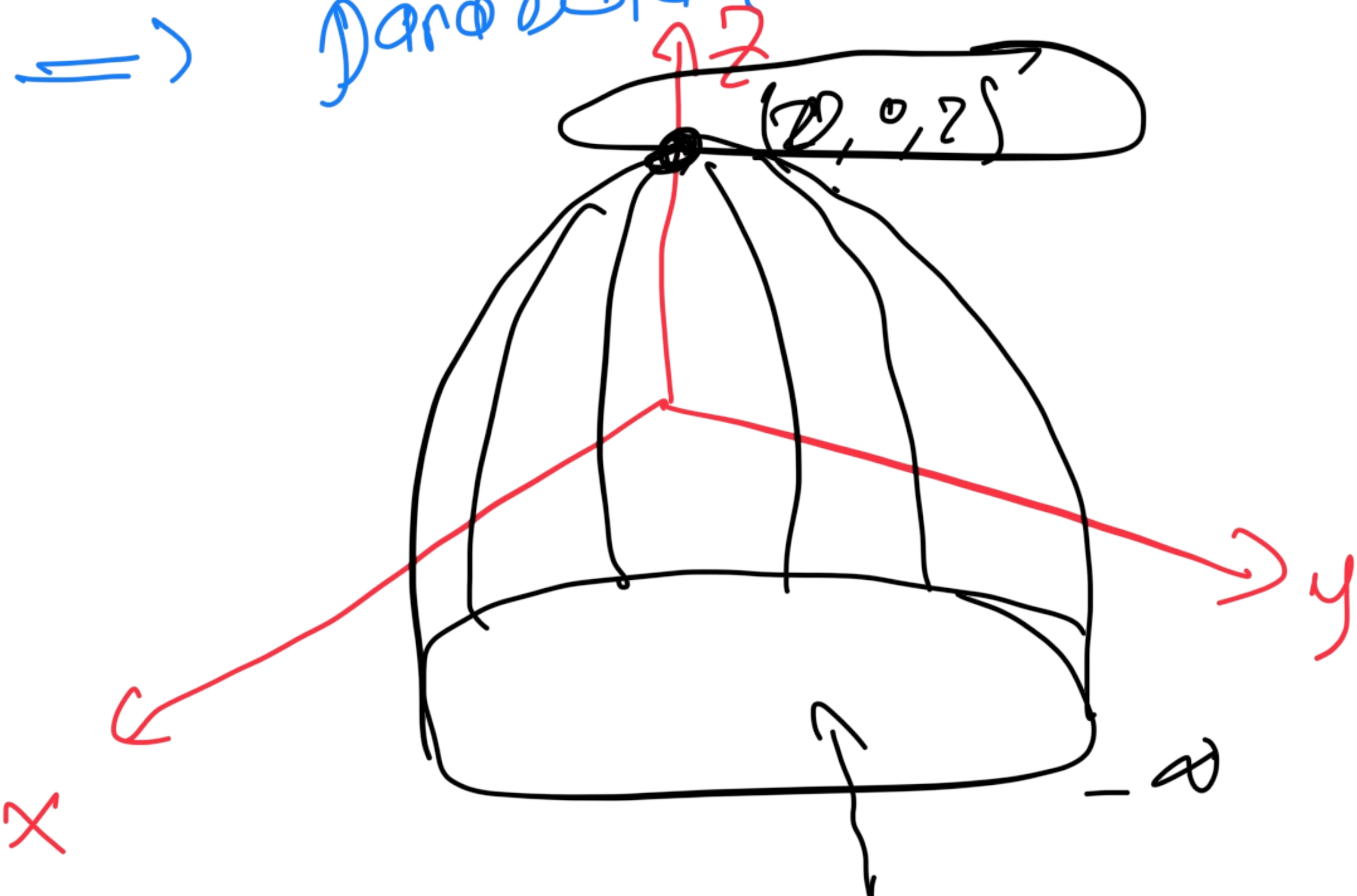
parabola

Q28 $f(x,y) = 2 - x^2 - y^2$

$z = 2 - x^2 - y^2$

$z - 2 = -x^2 - y^2$

$\Rightarrow$ paraboloid circular



Range $[-\infty, 2]$

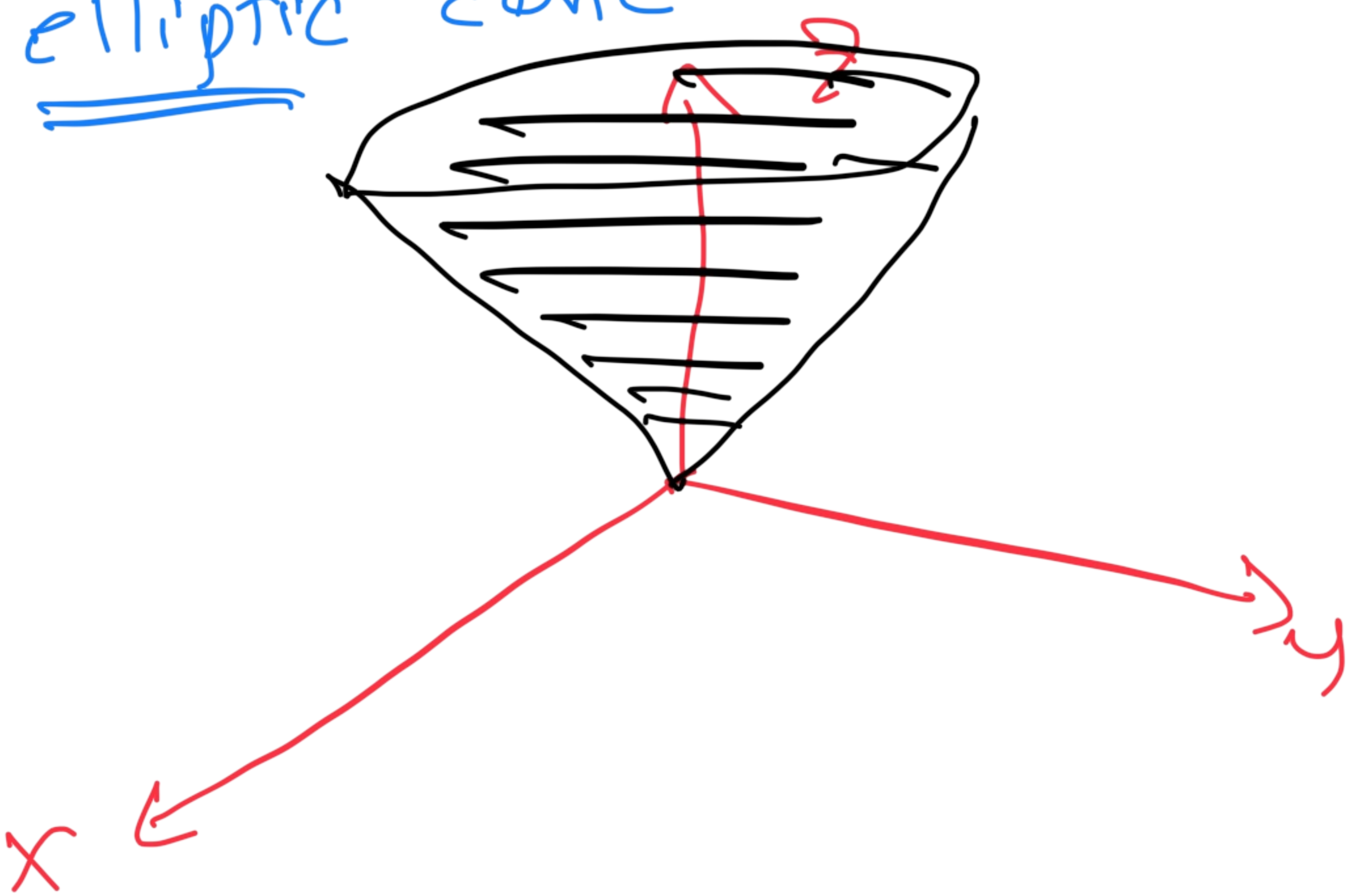
3

Q30 $f(x,y) = \sqrt{4x^2+y^2}$

$(z)^2 = \left(\sqrt{4x^2+y^2}\right)^2$

$$\boxed{z^2 = 4x^2 + 1y^2}$$ → cone

elliptic cone   " half "



Q31 $f(x,y) = \sqrt{4 - 4x^2 - y^2}$

$z = $

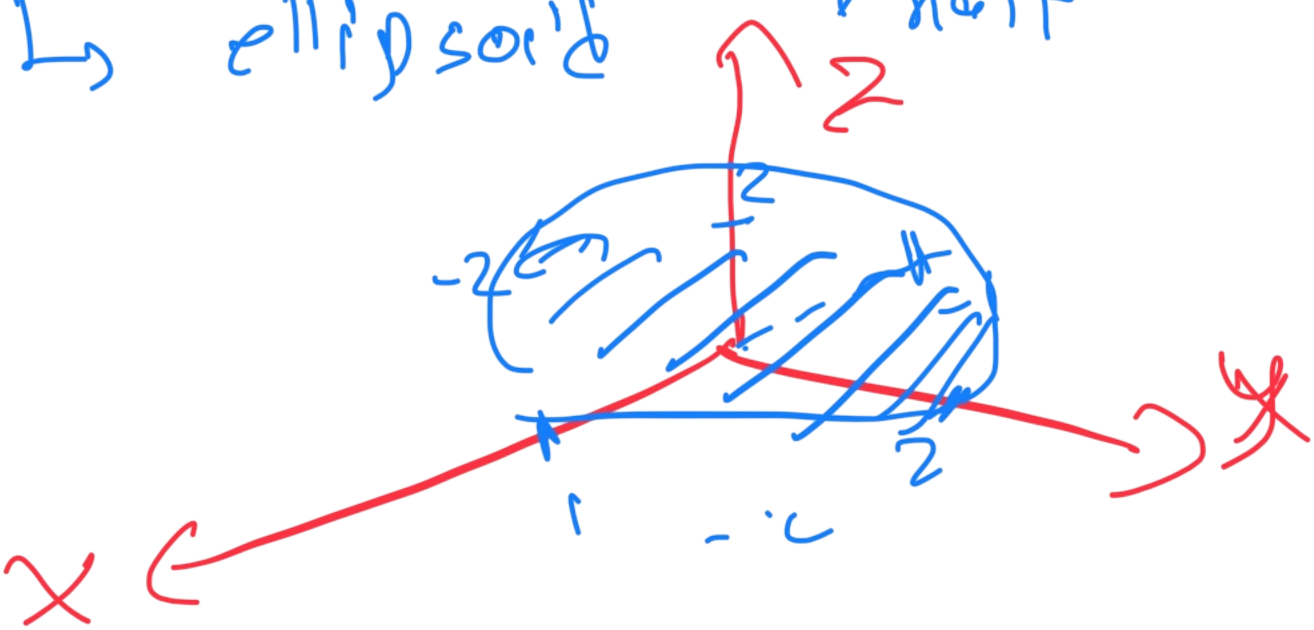
$(z)^2 = (\sqrt{4 - 4x^2 - y^2})^2$

$z^2 = 4 - 4x^2 - y^2$

$\Rightarrow \frac{4x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} = \frac{4}{4}$

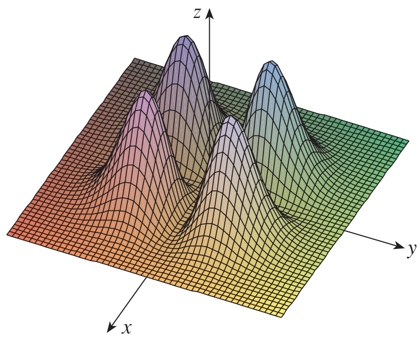
$\Rightarrow \boxed{\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1}$

↳ ellipsoid "half"



37. Locate the points A and B on the map of Lonesome Mountain (Figure 12). How would you describe the terrain near A? Near B?

38. Make a rough sketch of a contour map for the function whose graph is shown.



39. The *body mass index* (BMI) of a person is defined by

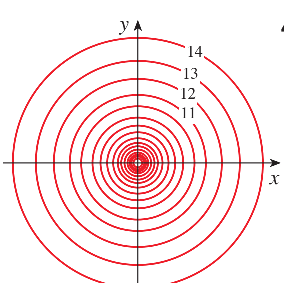
$$B(m, h) = \frac{m}{h^2}$$

where m is the person's mass (in kilograms) and h is the person's height (in meters). Draw the level curves $B(m, h) = 18.5$, $B(m, h) = 25$, $B(m, h) = 30$, and $B(m, h) = 40$. A rough guideline is that a person is underweight if the BMI is less than 18.5; optimal if the BMI lies between 18.5 and 25; overweight if the BMI lies between 25 and 30; and obese if the BMI exceeds 30. Shade the region corresponding to optimal BMI. Does someone who weighs 62 kg and is 152 cm tall fall into the optimal category?

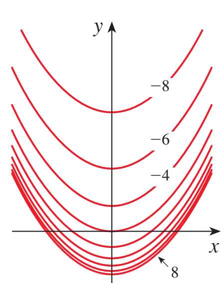
40. The body mass index is defined in Exercise 39. Draw the level curve of this function corresponding to someone who is 200 cm tall and weighs 80 kg. Find the weights and heights of two other people with that same level curve.

41–44 A contour map of a function is shown. Use it to make a rough sketch of the graph of f.

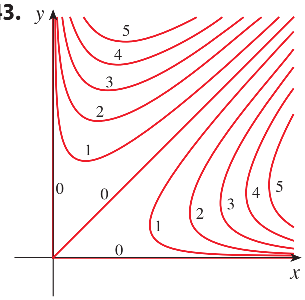
41.



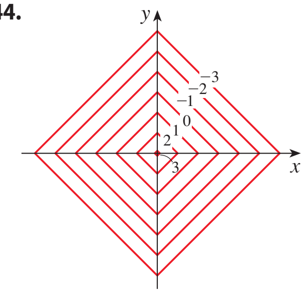
42.



43.



44.



45–52 Draw a contour map of the function showing several level curves.

45. $f(x, y) = x^2 - y^2$

46. $f(x, y) = xy$

47. $f(x, y) = \sqrt{x} + y$

48. $f(x, y) = \ln(x^2 + 4y^2)$

49. $f(x, y) = ye^x$

50. $f(x, y) = y - \arctan x$

51. $f(x, y) = \sqrt[3]{x^2 + y^2}$

52. $f(x, y) = y/(x^2 + y^2)$

53–54 Sketch both a contour map and a graph of the given function and compare them.

53. $f(x, y) = x^2 + 9y^2$

54. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

55. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). Sketch some level curves (isothermals) if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}$$

56. If $V(x, y)$ is the electric potential at a point (x, y) in the xy-plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

57–60 Graph the function using various domains and viewpoints. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

57. $f(x, y) = xy^2 - x^3$ (monkey saddle)

58. $f(x, y) = xy^3 - yx^3$ (dog saddle)

59. $f(x, y) = e^{-(x^2+y^2)/3}(\sin(x^2) + \cos(y^2))$

60. $f(x, y) = \cos x \cos y$

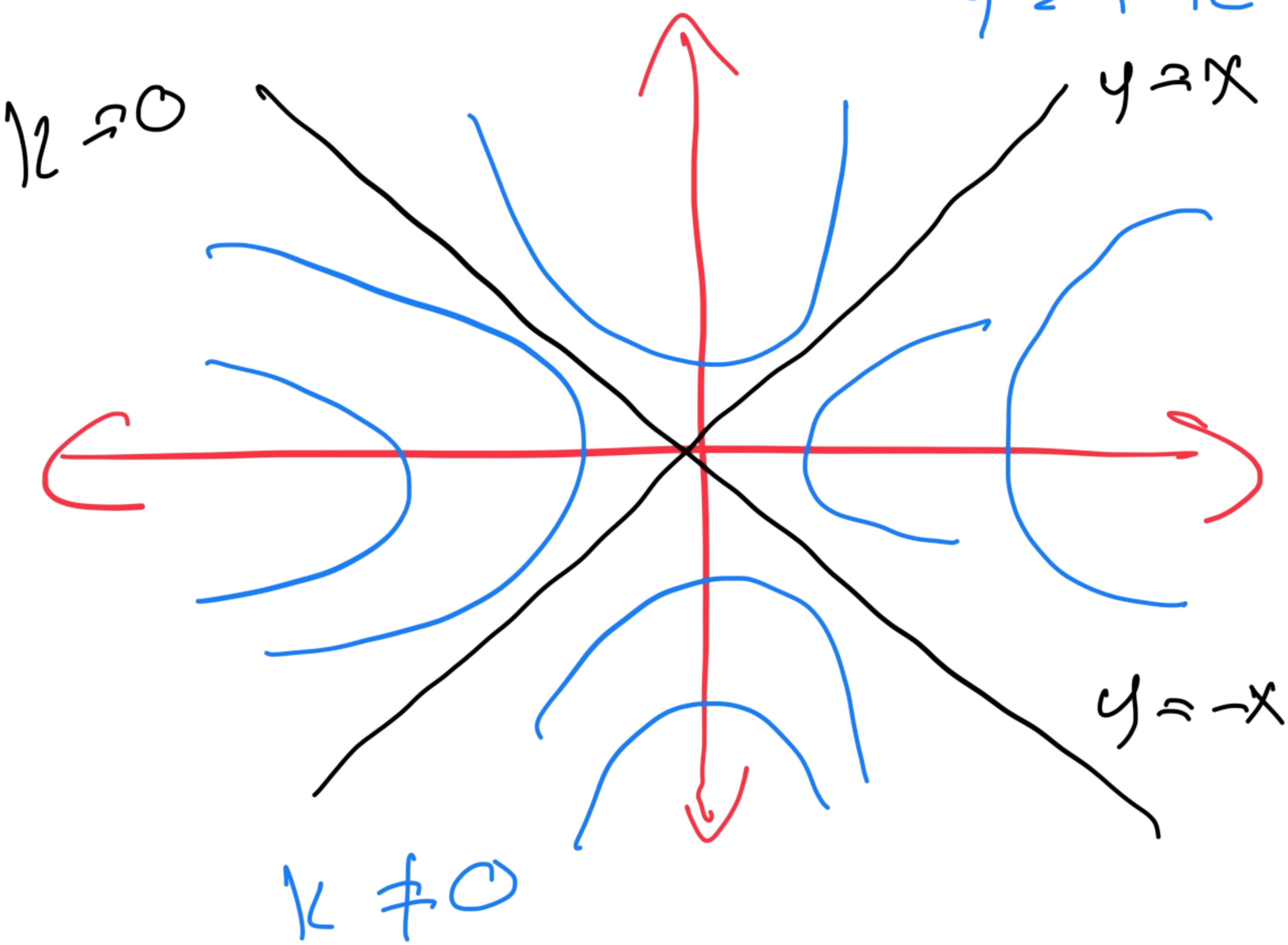
Q45 $f(x,y) = x^2 - y^2$

$k = x^2 - y^2$

$k = 0 \Rightarrow x^2 - y^2 = 0$

$\sqrt{x^2} = \sqrt{y^2} \rightarrow x = \pm y$

$y = \pm k$



$k \neq 0$

$x^2 - y^2 = 5$

$= 6$

$= -1$

$= 7$

hyperbolas

Q) $f(x,y) = \ln(x^2+4y^2)$

48

$e^k = \cancel{e}^{\cancel{\ln}(x^2+4y^2)}$ $\boxed{A = e^k}$

$\rightarrow \boxed{x^2+4y^2 = A}$

ellipsoid

$k = 0$ $k = +$ $k = -$