2. Use a graphing calculator or computer to draw the graphs of hypocycloids with a a positive integer and $b = 1$. How does the value of a affect the graph? Show that if we take $a = 4$, then the parametric equations of the hypocycloid reduce to

$$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

This curve is called a **hypocycloid of four cusps**, or an **astroid**.

3. Now try $b = 1$ and $a = n/d$, a fraction where n and d have no common factor. First let $n = 1$ and try to determine graphically the effect of the denominator d on the shape of the graph. Then let n vary while keeping d constant. What happens when $n = d + 1$?

4. What happens if $b = 1$ and a is irrational? Experiment with an irrational number like $\sqrt{2}$ or $e - 2$. Take larger and larger values for θ and speculate on what would happen if we were to graph the hypocycloid for all real values of θ.

5. If the circle C rolls on the *outside* of the fixed circle, the curve traced out by P is called an **epicycloid**. Find parametric equations for the epicycloid.

6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2–4.

10.2 Calculus with Parametric Curves

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, areas, arc length, speed, and surface area.

Ex

Tangents

مشتقة

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

If we think of the curve as being traced out by a moving particle, then dy/dt and dx/dt are the vertical and horizontal velocities of the particle and Formula 1 says that the slope of the tangent is the ratio of these velocities.

[illegible]

$$\boxed{1} \qquad \boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

Equation 1 (which you can remember by thinking of canceling the dt's) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t. We see from (1) that the curve has a horizontal tangent when $dy/dt = 0$, provided that $dx/dt \neq 0$, and it has a vertical tangent when $dx/dt = 0$, provided that $dy/dt \neq 0$. (If both $dx/dt = 0$ and $dy/dt = 0$, then we would need to use other methods to determine the slope of the tangent.) This information is useful for sketching parametric curves.

Parametric eq

$$\begin{tikzcd} \underline{\underline{y}} \arrow[r, bend left] & = f(\underline{t}) \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, bend right] & = g(t) \end{tikzcd}$$

Cart eq

$$y = f(x)$$

$$\frac{dy}{\underline{\underline{dx}}} = \frac{dy/dt}{dx/dt}$$

As we have learned, it is also often useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in Equation 1:

Note that $\dfrac{d^2y}{dx^2} \neq \dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}}$$

EXAMPLE 1 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.
(b) Find the points on C where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

SOLUTION

(a) Notice that $x = 3$ for $t = \pm\sqrt{3}$ and, in both cases, $y = t(t^2 - 3) = 0$. Therefore the point $(3, 0)$ on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that C crosses itself at $(3, 0)$. Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

the slope of the tangent when $t = \sqrt{3}$ is $dy/dx = 6/(2\sqrt{3}) = \sqrt{3}$, and when $t = -\sqrt{3}$ the slope is $dy/dx = -6/(2\sqrt{3}) = -\sqrt{3}$. Thus we have two different tangent lines at $(3, 0)$ with equations

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

(b) C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$. The corresponding points on C are $(1, -2)$ and $(1, 2)$. C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.) The corresponding point on C is $(0, 0)$.

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}\left(\dfrac{3t^2 - 3}{2t}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{6t^2 + 6}{4t^2}}{2t} = \frac{3t^2 + 3}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch C in Figure 1. ■

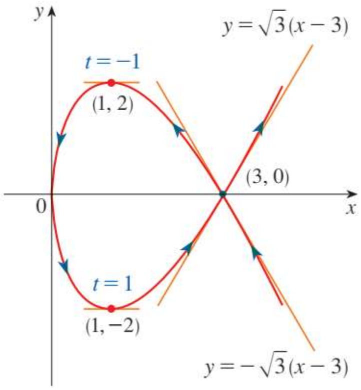

FIGURE 1

EXAMPLE 2

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$. (See Example 10.1.9.)
(b) At what points is the tangent horizontal? When is it vertical?

$$\frac{\partial y}{\partial x} \longrightarrow \boxed{\frac{\partial y / \partial t}{\partial x / \partial t}}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial / \partial t \left(\partial y / \partial x \right)}{\partial x / \partial t}$$

Ex ①

$$x \leftarrow \overset{x \quad y}{(3, 0)} \rightarrow y$$

$$x = t^2 \qquad y = t^3 - 3t$$

a) $t^2 = 3$

$$\boxed{t = \mp\sqrt{3}}$$

$$t^3 - 3t = 0$$

$$t(t^2 - 3) = 0$$

$$\times \; t = 0 , \quad \boxed{t = \mp\sqrt{3}}$$

$$t = \underline{\pm\sqrt{3}} \longleftarrow \longrightarrow \underline{(\overset{x_1}{3}, \overset{y_1}{0})}$$

$$t = 0 \longrightarrow (0, 0) \quad X$$

tangent line equation

$$y - y_1 = \underline{\underline{m}} \, (x - x_1)$$

$$=$$

$$m = \frac{dy}{dx} = \underline{\underline{\frac{dy/dt}{dx/dt}}}$$

$$\swarrow$$

$$m = \frac{3t^2 - 3}{2t} \longleftarrow$$

$$t = \sqrt{3} \longrightarrow m = \sqrt{3}$$

$$t = -\sqrt{3} \longrightarrow m = \frac{3(-\sqrt{3})^{2} - 3}{2(-\sqrt{3})}$$

$$\hookrightarrow m = \frac{6}{-\cancel{2}\sqrt{3}} = \frac{-3}{\sqrt{3}} = \boxed{-\sqrt{3}}$$

$$(3, \underline{\underline{0}})$$

$$m = \sqrt{3}$$

$$m = -\sqrt{3}$$

$$\boxed{\begin{aligned} y - 0 &= \sqrt{3}(x-3) \\ y - 0 &= -\sqrt{3}(x-3) \end{aligned}}$$

b) horizontal → بسط = صفر
مقام ≠ صفر

$$\frac{\partial y}{\partial x} = \frac{3t^2 - 3}{2t}$$

$$\cancel{3}t^2 = \cancel{3}1$$

$$\hookrightarrow \frac{\partial y}{\partial t} = 3t^2 - 3 = 0 \rightarrow \boxed{t = \mp 1}$$

مقام $\frac{\partial x}{\partial t} = 2t \neq 0 \rightarrow \boxed{t \neq 0}$

Vertical → مقام = صفر
بسط ≠ صفر

$$\frac{\partial y}{\partial t} = 3t^2 - 3 \neq 0 \Rightarrow t \neq \mp 1$$

$$\frac{\partial x}{\partial t} = 2t = 0 \rightarrow \boxed{t = 0}$$

$$\boxed{\begin{array}{ll} \text{horz} & t = \pm 1 \\ \text{vert} & t = 0 \end{array}}$$

concave up / down

$$\hookrightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

$$\underline{\underline{y}} = t^3 - 3t \rightarrow \frac{dy}{\underline{dt}} = \underline{\underline{3t^2 - 3}}$$

$$x = t^2 \rightarrow \frac{dx}{\underline{\underline{dt}}} = 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}$$

$$\frac{d^2y}{dx^2} = \boxed{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}$$

$$2t$$

$$12t^2 \qquad - \qquad -6t^2+6$$

$$\frac{d^2y}{dx^2} = \frac{(6t)(2t) - (3t^2-3)(2)}{(2t)^{2}}$$

$$4t^2 \qquad 2t$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{6t^2+6/2}{8t^3}}$$

$$/2$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{3t^2+3}{4t^3} \neq 0} = 0$$

∴ مرفوض

$t = 0$

$t > 0 \longrightarrow$ con up +

$t < 0 \longrightarrow$ − con down

$$\frac{d^2y}{dx^2} = \frac{+}{\quad}$$

Ex ②   tangent → $\theta \to \pi/3$

$$x = r(\theta - \sin\theta) \qquad r : \text{[illegible]}$$

$$y = r(\underset{0}{1} - \underset{+\,\sin\theta}{\cos\theta})$$

a) tangent → $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

$$\frac{dy}{dx} = \frac{\cancel{r}\,\sin\theta}{\cancel{r}\,[1 - \cos\theta]}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \underset{m}{\boxed{\sqrt{3}}}$$

$$y - \underline{\underline{y_1}} = \underline{\underline{m}}\,(x - \underline{\underline{x_1}})$$

$$y_1 = \underline{\underline{r}}\left(1 - \overset{1/2}{\underline{\underline{\cos \pi/3}}}\right) = \boxed{\frac{r}{2}}$$

$$x_1 = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) =$$

$$x_1 = r\left[\frac{\pi}{2} - \frac{\sqrt{3}}{2}\right)$$

$$\boxed{y - \frac{r}{2} = \sqrt{3}\left[x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]}$$

b) $$\frac{\partial y}{\partial x} = \frac{\sin\theta}{1 - \cos\theta}$$

horizontal بيساوي صفر

$\sin\theta = 0 \rightarrow$

عند $\theta = 0, \pi, 2\pi, 3\pi, \ldots$

$1 - \cos\theta \neq 0 \quad , \quad 1 \neq \cos\theta$

$\cos\theta \neq 1$ مقام ≠ صفر

ابعاد $\theta = 0, 2\pi, 4\pi, 6\pi .$

نوصل

$$\boxed{\text{hor } \theta = (2n-1)\pi}$$

Even $n \rightarrow 2n \Rightarrow$ (عدد زوجي)

SOLUTION

(a) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and
$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-\frac{1}{2}} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \qquad \text{or} \qquad \sqrt{3}x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$

The tangent is sketched in Figure 2.

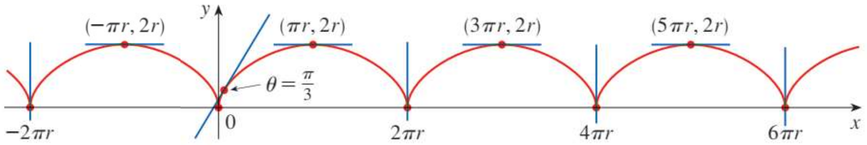

FIGURE 2

(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $\sin\theta = 0$ and $1 - \cos\theta \neq 0$, that is, $\theta = (2n-1)\pi$, n an integer. The corresponding point on the cycloid is $((2n-1)\pi r, 2r)$.

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta\to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta\to 2n\pi^+} \frac{\sin\theta}{1-\cos\theta} = \lim_{\theta\to 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^-$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$. (See Figure 2.) ■

■ Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

The limits of integration for t are found as usual with the Substitution Rule. When $x = a$, t is either α or β. When $x = b$, t is the remaining value.

EXAMPLE 3 Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

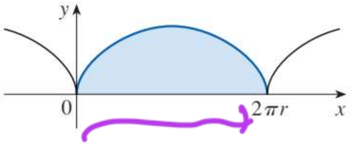

FIGURE 3

The result of Example 3 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (see Example 10.1.9). Galileo guessed this result but it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.

SOLUTION One arch of the cycloid (shown in Figure 3) is given by $0 \leqslant \theta \leqslant 2\pi$. Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$, we have

$$\begin{aligned} A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta \\ &= r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = r^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= r^2\int_0^{2\pi}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\ &= r^2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= r^2\left(\tfrac{3}{2}\cdot 2\pi\right) = 3\pi r^2 \end{aligned}$$

■

■ Arc Length

طول القوس

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 8.1.3 says that if F' is continuous, then

$$\boxed{2} \qquad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt$$

Since $dx/dt > 0$, we have

$$\boxed{3} \qquad L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

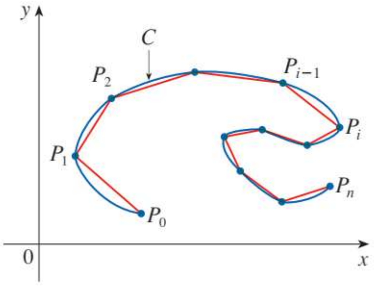

FIGURE 4

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygonal path with vertices $P_0, P_1, \ldots, P_n$ approximates C. (See Figure 4.)

As in Section 8.1, we define the length L of C to be the limit of the lengths of these approximating polygonal paths as $n \to \infty$:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$. Then the preceding equation becomes

$$\Delta x_i = f'(t_i^*)\,\Delta t$$

Ex ③

$$\underline{\underline{x}} = r(\underline{\theta} - \underline{\underline{\sin\theta}})$$

$$\underline{\underline{y}} = r(1 - \cos\theta) \leftarrow \leftarrow$$

$$\boxed{A = \int_{a=0}^{b=2\pi r} \underline{\underline{f(x)}}^{y} \, \underline{\underline{dx}}} \quad f(x) = y$$

$$A = \int_{0}^{2\pi r} \underline{\underline{r}}(1 - \underline{\underline{\cos\theta}}) \, \underline{\underline{r}}(1 - \underline{\underline{\cos\theta}}) d\theta$$

$$A = r^2 \int_{0}^{2\pi r} (1 - \cos\theta)^{②} d\theta$$

$$r^2 \int_{0} 1 - 2\cos\theta + \cos^2\theta$$

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$A = r^2 \int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \cancel{\frac{1}{2}} + \frac{1}{2}\cos 2\theta$$

$$A = r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \underset{0}{\frac{\sin 2\theta}{4}} \right]_0^{\boxed{2\pi}}$$

$$A = \boxed{r^2}\, \frac{3 \cdot \cancel{2}\pi}{\cancel{2}}$$

$\sin 2\pi = 0$

$$\boxed{A = 3\pi r^2}$$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**})\,\Delta t$$

Therefore

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\,\Delta t]^2 + [g'(t_i^{**})\,\Delta t]^2} \\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t \end{aligned}$$

and so

$$\boxed{4} \qquad L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t$$

The sum in (4) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if f' and g' are continuous, it can be shown that the limit in (4) is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\; dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.

5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

Notice that the formula in Theorem 5 is consistent with the general formula $L = \int ds$ of Section 8.1, where

$$\boxed{6} \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

EXAMPLE 4 If we use the representation of the unit circle given in Example 10.1.2,

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so Theorem 5 gives

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\; dt = \int_0^{2\pi} dt = 2\pi$$

as expected. If, on the other hand, we use the representation given in Example 10.1.3,

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = 2\cos 2t$, $dy/dt = -2\sin 2t$, and the integral in Theorem 5 gives

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt = \int_0^{2\pi} \sqrt{4\cos^2(2t) + 4\sin^2(2t)}\; dt = \int_0^{2\pi} 2\; dt = 4\pi$$

Ex ④

$x = \cos t \qquad \overset{a}{0} \le t \le \overset{b}{2\pi}$

$y = \sin t$

$$L = \int_a^b \sqrt{(y')^2 + (x')^2}\, dt$$

$$L = \int_0^{2\pi} \sqrt{\underbrace{(\cos t)^2 + (-\sin t)^2}_{\cos^2 t + \sin^2 t = 1}}\, dt \qquad \sqrt{1}$$

$y' = \cos t$

$x' = -\sin t$

$$L = \int_0^{2\pi} 1\, dt = t\Big]_0^{2\pi}$$

$$\boxed{L = 2\pi}$$

$x = \sin 2t$

$$\boxed{x' = 2\cos 2t}$$

$0 \le t \le 2\pi$

$y = \cos 2t$

$$\boxed{y' = -2\sin 2t}$$

$$L = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}$$

$$4[\underbrace{\cos^2 + \sin^2 t}_{①}]$$

$$= \int \sqrt{4} = \int 2 = 2(2\pi - 0)$$

$$= \boxed{4\pi}$$

Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β. ■

EXAMPLE 5 Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

SOLUTION From Example 3 we see that one arch is described by the parameter interval $0 \leqslant \theta \leqslant 2\pi$. Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

The result of Example 5 says that the length of one arch of a cycloid is eight times the radius of the generating circle (see Figure 5). This was first proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul's Cathedral in London.

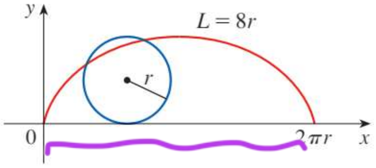

FIGURE 5

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos\theta = 2\sin^2(\theta/2)$. Since $0 \leqslant \theta \leqslant 2\pi$, we have $0 \leqslant \theta/2 \leqslant \pi$ and so $\sin(\theta/2) \geqslant 0$. Therefore

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\left|\sin(\theta/2)\right| = 2\sin(\theta/2)$$

and so

$$L = 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi}$$

$$= 2r[2 + 2] = 8r$$ ■

The arc length function and speed

Recall that the arc length function (Formula 8.1.5) gives the length of a curve from an initial point to any other point on the curve. For a parametric curve C given by $x = f(t)$, $y = g(t)$, where f' and g' are continuous, we let $s(t)$ be the arc length along C from an initial point $(f(\alpha), g(\alpha))$ to a point $(f(t), g(t))$ on C. By Theorem 5, the **arc length function** s for parametric curves is

position

7 $$s(t) = \int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

(We have replaced the variable of integration by u so that t does not have two meanings.)

If parametric equations describe the position of a moving particle (with t representing time), then the **speed** of the particle at time t, $v(t)$, is the rate of change of distance traveled (arc length) with respect to time: $s'(t)$. By Equation 7 and Part 1 of the Fundamental Theorem of Calculus, we have

velocity

8 $$\boxed{v(t) = s'(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

Ex ⑤

$$x = r(\theta - \sin\theta)$$

$$\boxed{x' = r[1 - \cos\theta]} \quad 0 \leq \theta \leq 2\pi$$

$$y = r(1 - \cos\theta)$$

$$\boxed{y' = r[\sin\theta]}$$

$$L = \int_a^b \sqrt{y'^2 + x'^2}\, d\theta$$

$$L = \int_0^{2\pi} \sqrt{r^2 \sin^2\theta + \underline{\underline{r^2}}(1 - \cos\theta)^2}$$

$$= \sqrt{r^2[\sin^2\theta + (1 - \cos\theta)^2}$$

$$\int r^2 \sqrt{\quad}$$

$$L = r\int_{0}^{2\pi} \sqrt{\underline{\sin^2\theta} + \underline{1} - 2\cos\theta + \underline{\cos^2\theta}}$$

$$L = r\int_{0}^{2\pi} \sqrt{\underline{1+1} - 2\cos\theta}$$

$$2 - 2\cos\theta$$

$$\sqrt{2}\sqrt{(1 - \cos\theta)}$$

$$L = \sqrt{2}\, r\int_{0}^{2\pi} \sqrt{\underline{1 - \cos\theta}}\, d\theta$$

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\rightarrow \left(\sin^2\theta = \frac{1}{\underline{\underline{2}}} - \frac{1}{2}\cos 2\theta\right)\underline{\underline{2}}$$

$$\underline{\underline{2\sin^2\theta}} = \underline{\underline{1 - \cos \boxed{\underline{\underline{2\theta}}}}}$$

$$L = \underline{\underline{\sqrt{2}r}} \int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta$$

$$\underline{\underline{\sqrt{2}}} \quad \cancel{\sqrt{\ }}\,\sin\theta$$

$$L = \underline{\underline{2r}} \int_0^{2\pi} \sin\boxed{\left(\frac{\theta}{2}\right)}\, d\theta$$

$$L = 2r \left[\frac{-\cos\left(\frac{\theta}{2}\right)}{1/2} \right]_0^{2\pi}$$

$$\boxed{L = 8r}$$

EXAMPLE 6 The position of a particle at time t is given by the parametric equations $x = 2t + 3$, $y = 4t^2$, $t \geqslant 0$. Find the speed of the particle when it is at the point $(5, 4)$.

SOLUTION By Equation 8, the speed of the particle at any time t is

$$v(t) = \sqrt{2^2 + (8t)^2} = 2\sqrt{1 + 16t^2}$$

The particle is at the point $(5, 4)$ when $t = 1$, so its speed at that point is $v(1) = 2\sqrt{17} \approx 8.25$. (If distance is measured in meters and time in seconds, then the speed is approximately 8.25 m/s.) ■

مساحة السطح

■ Surface Area

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. Suppose a curve C is given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f', g' are continuous, $g(t) \geqslant 0$, and C is traversed exactly once as t increases from α to β. If C is rotated about the x-axis, then the area of the resulting surface is given by

$$\boxed{9} \qquad S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, where ds is given by Formula 6.

EXAMPLE 7 Show that the surface area of a sphere of radius r is $4\pi r^2$.

SOLUTION The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \leqslant t \leqslant \pi$$

about the x-axis. Therefore, from Formula 9, we get

$$S = \int_0^{\pi} 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= 2\pi \int_0^{\pi} r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt = 2\pi \int_0^{\pi} r \sin t \cdot r\, dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin t\, dt = 2\pi r^2(-\cos t)\Big]_0^{\pi} = 4\pi r^2$$ ■

$S = 4\pi r^2$

10.2 Exercises

1–4 Find dx/dt, dy/dt, and dy/dx.

1. $x = 2t^3 + 3t$, $y = 4t - 5t^2$
2. $x = t - \ln t$, $y = t^2 - t^{-2}$
3. $x = te^t$, $y = t + \sin t$
4. $x = t + \sin(t^2 + 2)$, $y = \tan(t^2 + 2)$

Ex 6

$$x = \dot{2t} + \dot{3} = 5 \rightarrow \boxed{t = 1}$$

$$y = \underline{\underline{4t^2}} = 4 \leftarrow \boxed{t = 1}$$

speed at $\underline{\underline{(5, 4)}}$

x y

$$\boxed{v(t=1)}$$

$$v = s'(t) = \sqrt{y'^2 + x'^2}$$

$$v(t) = \sqrt{(8t)^2 + (2)^2}$$

8^2

$$v(t=1) = \sqrt{8^2 + 2^2}$$

$$v(1) = \sqrt{68} = \boxed{2\sqrt{17}}$$

Ex ⑦

$$x = r\cos t \qquad \overset{a}{\underline{0}} \le t \le \overset{b}{\underline{\pi}}$$

$$y = \underbrace{r\sin t}$$

$$S = \int_a^b 2\pi \underline{y} \sqrt{y'^2 + x'^2}\, dt$$

$$S = \int_0^{\pi} 2\pi\, r \sin t \sqrt{r^2\cos^2 t + r^2 \sin^2 t}$$

$$= \sqrt{r^2}\,(\underbrace{\cos^2 + \sin^2}_{1})$$

$$S = \int_0^{\pi} 2\pi r^2 \sin t \; dt$$

$$2\pi r^2 \left[-\cos t\right]_0^{\pi}$$

$$2\pi r^2 \left[-(-1)) - (-1))\right]$$

$$1 + 1 = 2$$

$$\boxed{S = 4\pi r^2}$$

10,2