

2.2 Axioms, interpretations, and Properties of Probability

In probability theory, the objective is to assign a probability $P(A)$ to each event A in a sample space S , which measures the likelihood of event A occurring. To ensure these probability assignments align with our intuitive understanding, they must satisfy three axioms (basic properties) of probability.

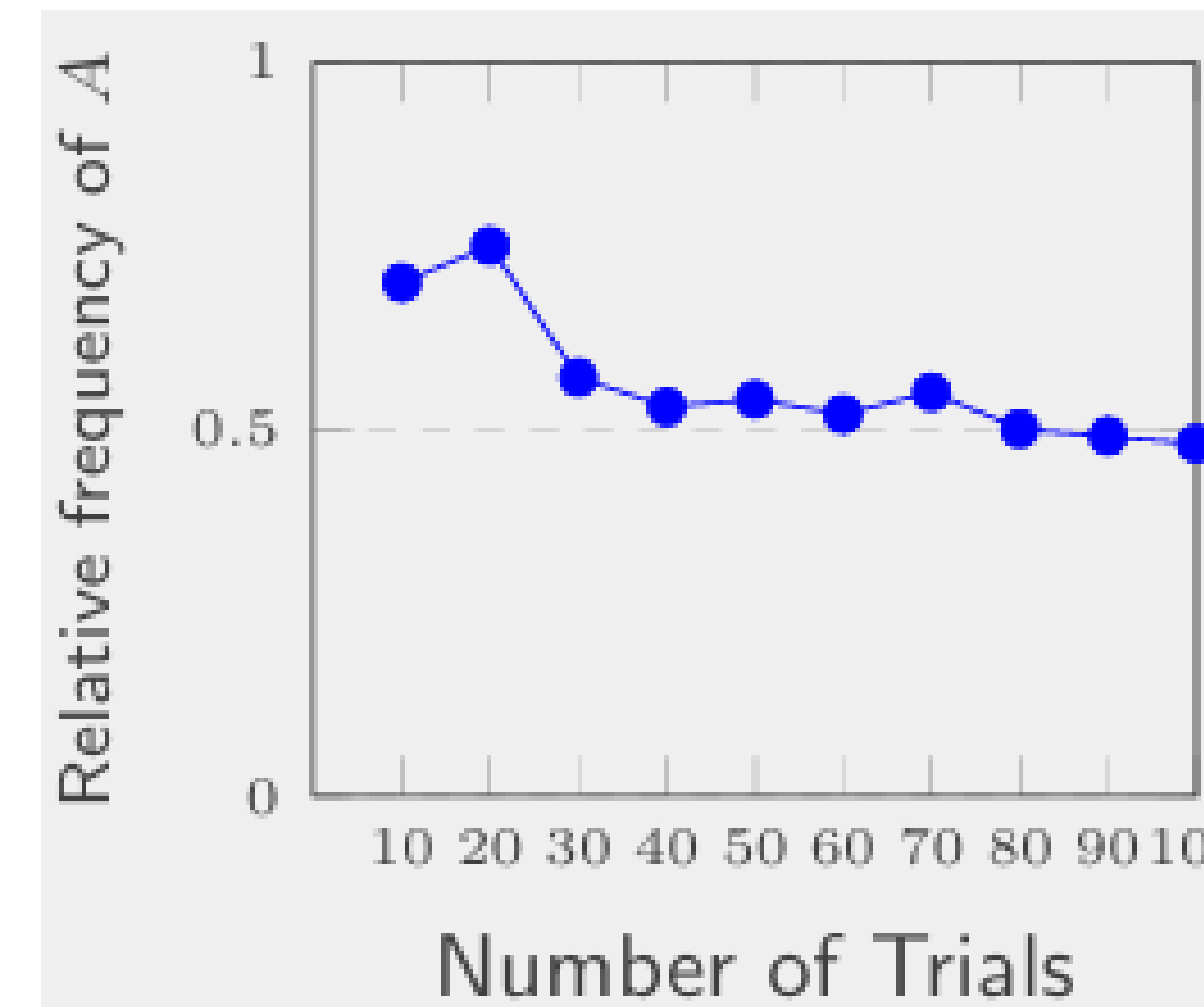
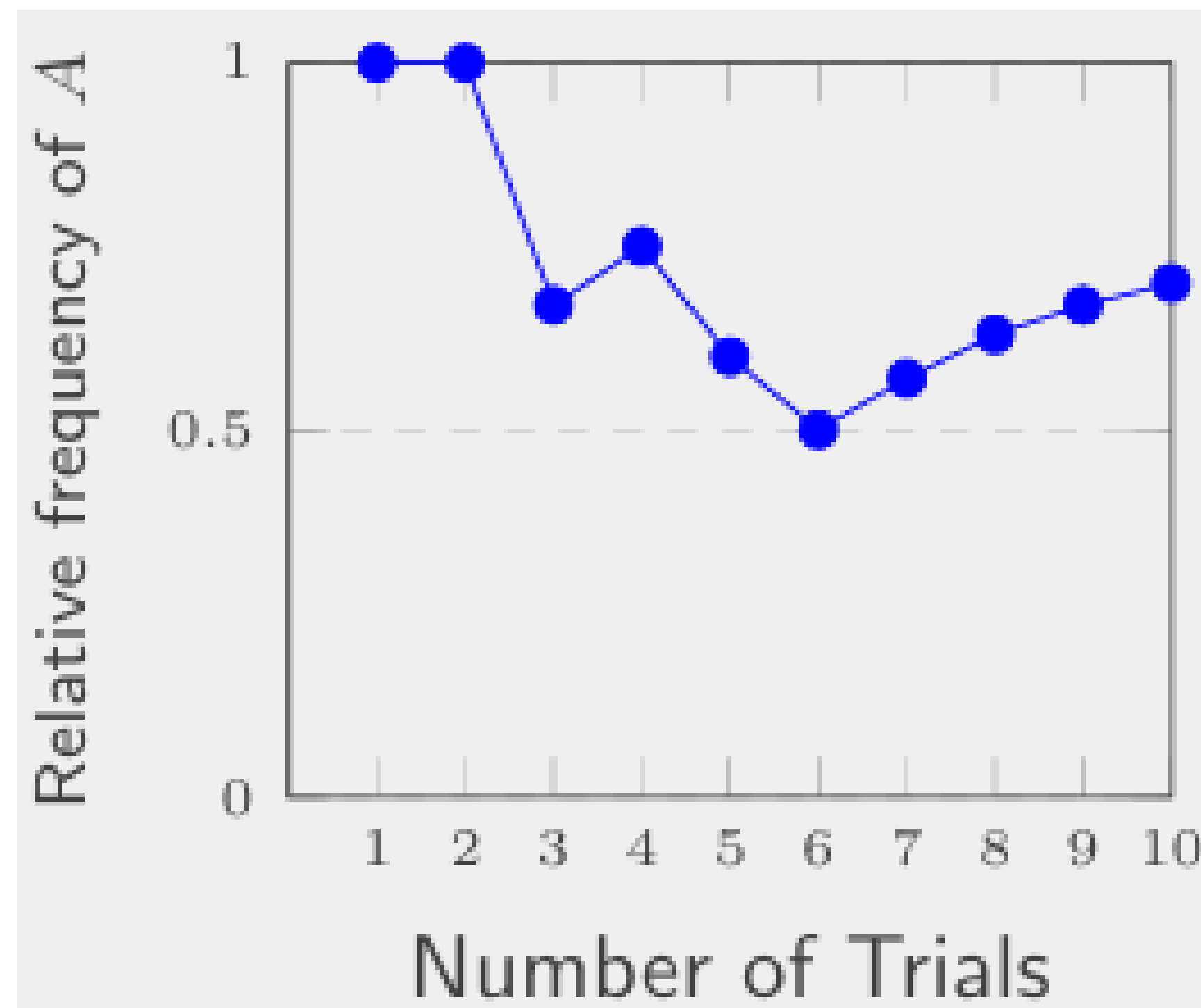
Probability is a measure used to quantify the likelihood (or chance) that an outcome of an experiment will occur.

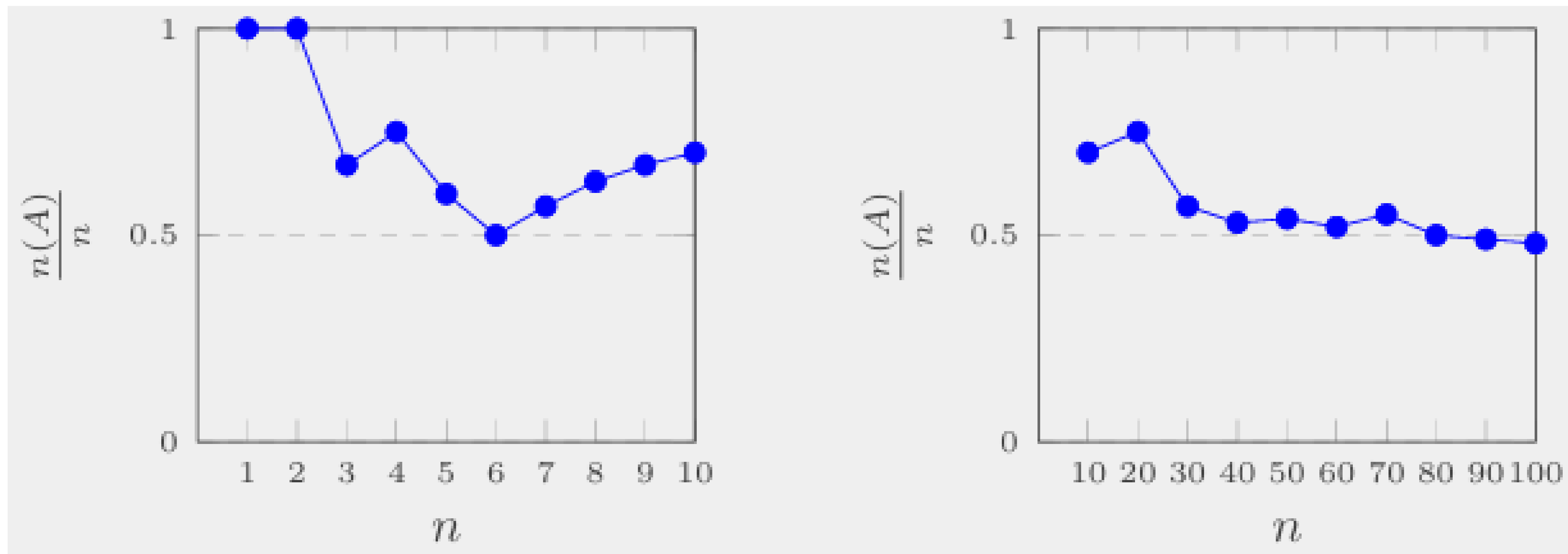
$$P(A) = \frac{\text{no. of } A}{\text{Total}}$$

➤ Relative Frequencies and Objective Probability

Let A be the event of a coin landing a “head” when tossed. Repeating this experiment for n trials and recording the relative frequencies of A at each instance will result in a table similar the one below.

Trial #	10	20	30	40	50	60	70	80	90	100
Relative frequency of A	.70	.75	.57	.53	.54	.52	.50	.50	.49	.48





Let $n(A)$ denote the number of replications on which A does occur. Then $n(A)/n$ is the relative frequency of A after n trials. The conclusion is

$$\lim_{n \rightarrow \infty} \frac{n(A)}{n} = 0.5$$

➤ Axioms of Probability

AXIOM 1

For any event A , $P(A) \geq 0$. $\rightarrow 0 \leq p \leq 1$

AXIOM 2

$P(\mathcal{S}) = 1$.

AXIOM 3

If $A_1, A_2, A_3, \dots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = \sum_{i=1}^{\infty} P(A_i)$$

PROPOSITION



$$A \cap B = \phi$$
$$P(A \cap B) = P(\phi) = \text{zero}$$

$P(\emptyset) = 0$ where $\emptyset$ is the null event (the event containing no outcomes whatsoever). This in turn implies that the property contained in Axiom 3 is valid for a *finite* collection of disjoint events.

Example 1. In the coin-tossing experiment, define all events and then assign possible probabilities for each that are consistent with the three axioms.

$$S = \{H, T\}$$

1st event $E_1 = \{H\} \Rightarrow P(E_1) = \frac{1}{2}$

2nd event $E_2 = \{T\} \Rightarrow P(E_2) = \frac{1}{2}$

3rd event $E_3 = \emptyset \Rightarrow P(E_3) = 0$

$$\sum_{i=1}^n E_i$$

$$= P(E_1) + P(E_2) + P(E_3)$$

$$= \frac{1}{2} + \frac{1}{2} + 0$$

$$\boxed{= 1}$$

E-X

$$P(E_1) = 0.2 \quad , \quad P(E_2) = 0.5$$

$$P(E_3) = ?$$

$$P(E_1) + P(E_2) + P(E_3) = 1$$

$$0.2 + 0.5 + P(E_3) = 1$$

$$\boxed{P(E_3) = 0.3}$$

All events

E_1 & E_2 & E_3

Example 2 (Example 2.12). In Consider testing batteries coming off an assembly line one by one until one having a voltage within prescribed limits is found. The simple events are $E_1 = \{S\}$, $E_2 = \{FS\}$, $E_3 = \{FFS\}$, $E_4 = \{FFFS\}$, Suppose the probability of any particular battery being satisfactory is 0.99. Show that these simple events satisfy **Axiom 2**.

$$p(S) = 0.99$$

$$p(F) = 1 - 0.99 = 0.01 \rightarrow$$

Textbook

Study

Examples
2.11

➤ More Probability Properties

Properties of Probability

Let A, B, C be events of the sample space S . Then

1. $0 \leq P(A) \leq 1$

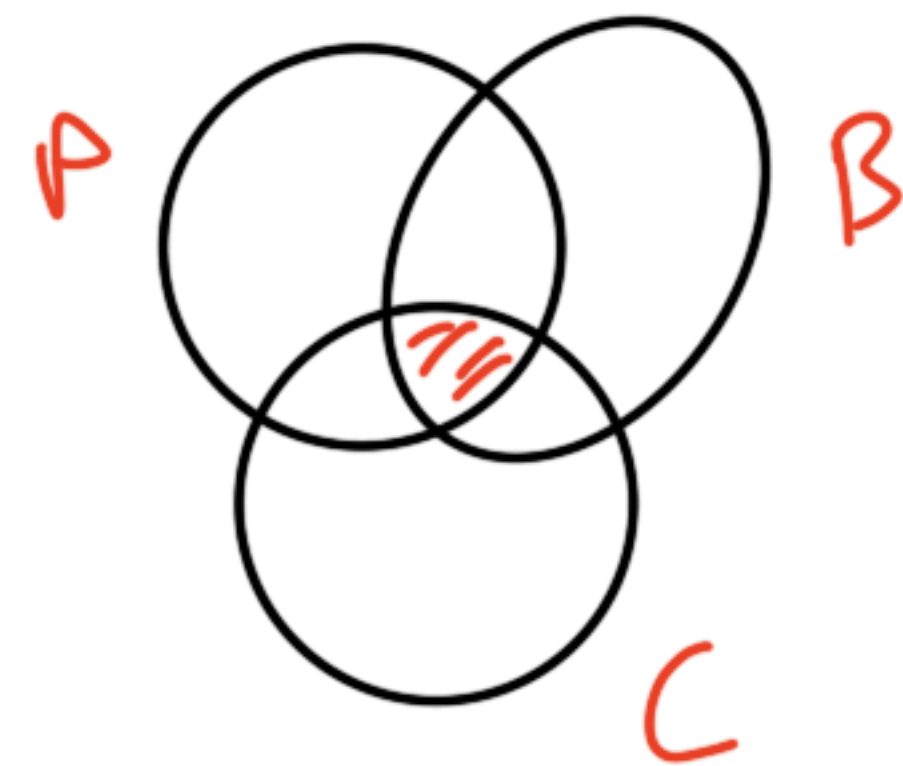
2. $P(\emptyset) = 0$

3. $P(A) + P(A') = 1 \rightarrow P(A') = 1 - P(A) \rightarrow P(A) = 1 - P(A')$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



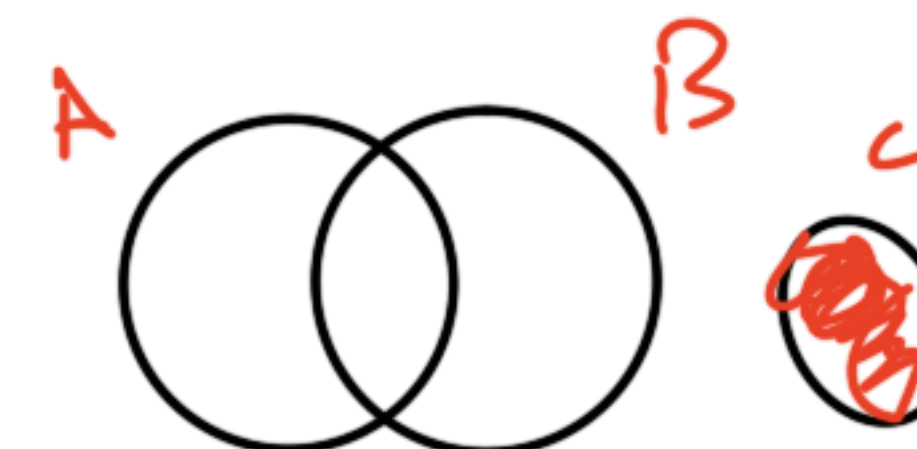
5. $P(A \cup B \cup C) = P(A) + P(B) + P(B) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$



De Morgan's Laws

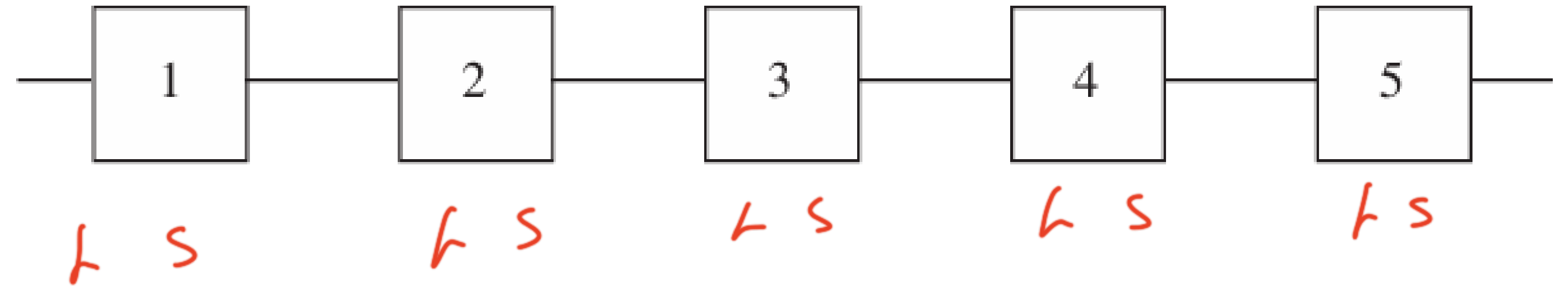
$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$



Example 3 (Example 2.13). Consider a system of five identical components connected in series, as illustrated below. 1 2 3 4 5 Denote a component that fails by F and one that does not by S . Define A as the event that the system fails with $P(A') = 0.59$.

- Find the probability that the system fails.
- How many outcomes are there in A ?
- How many outcomes are there in A' ?



$$P(A) + P(A') = 1$$

$$\textcircled{a} P(A) = 1 - P(A')$$

$$= 1 - 0.59 = 0.41$$

$$\text{Total outcomes} = 2^5 = 32$$

$A \rightarrow \text{fail}$

$A' \rightarrow \text{success}$ event $A' = \{SSSSS\}$

$$\textcircled{b} 32 - 1 = 31$$

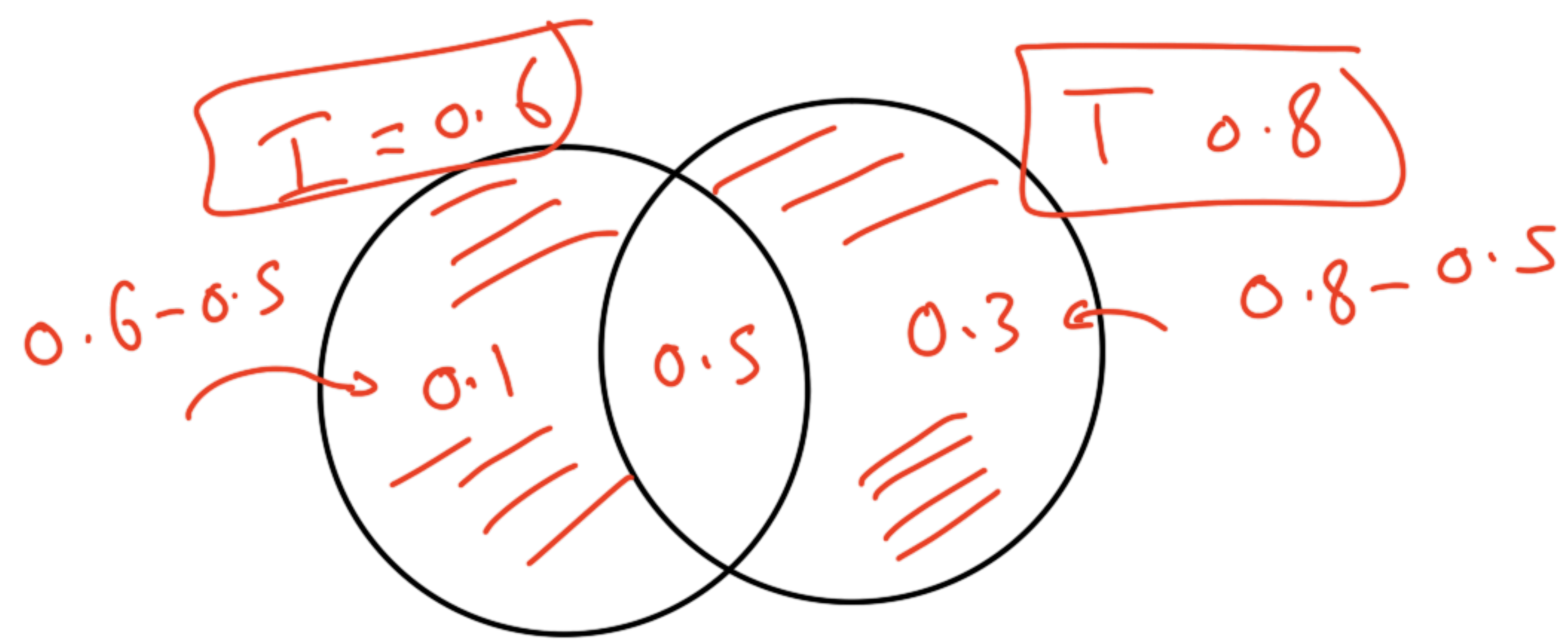
$$\textcircled{c} 1$$

Example 4 (Example 2.14). In a certain residential area, 60% of all households get Internet service from the local cable company, 80% get television service from that company, and 50% get both services from that company. If a household is randomly selected from this residential area, what is the probability that it gets:

- at least one of these two services from the company?
- exactly one of these services from the company?
- only TV service from the company?
- only Internet service from the company?

$$\begin{aligned}P(I) &= \underline{0.6} \\P(T) &= 0.8 \\ \text{both (and)} \quad P(I \cap T) &= 0.5\end{aligned}$$

$$\begin{aligned} \textcircled{a} \quad I \text{ or } T \quad (\text{at least}) \\ P(I \cup T) &= P(I) + P(T) - P(I \cap T) \\ &= 0.6 + 0.8 - 0.5 = 0.9 \end{aligned}$$



$$\begin{aligned}
 \textcircled{b} \quad & P(\text{exactly one}) \\
 &= 0.1 + 0.3 = 0.4 \\
 &= P(I \cup T) - P(I \cap T) \\
 &= 0.9 - 0.5 = 0.4
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{c} \quad \text{only T.V} &= P(T) - P(T \cap I) \\
 &= 0.8 - 0.5 = 0.3
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{d} \quad \text{only Internet} &= P(I) - P(T \cap I) \\
 &= 0.6 - 0.5 = 0.1
 \end{aligned}$$

Example 4 (Example 2.15). A commuter train has five cars. Suppose a commuter is twice as likely to select the middle car (#3) as to select either adjacent car (#2 or #4), and is twice as likely to select either adjacent car as to select either end car (#1 or #5).

- a. Find the probability that one of the three middle cars is selected. $p(2) + p(3) + p(4) = 0.2 + 0.4 + 0.2 = 0.8$
- b. What is the probability that car #1 is selected. $p(1) = 0.1$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ x & 2x & 4x & 2x & x \end{array}$$

$$p(1) + p(2) + p(3) + p(4) + p(5) = 10x = 1$$

$$x = \frac{1}{10}$$

$$p(1) = x = 0.1$$

$$p(2) = 2x = 0.2$$

$$p(3) = 4x = 0.4$$

$$p(4) = 2x = 0.2$$

$$p(5) = x = 0.1$$

➤ Equally Likely Assumption

The **equally likely assumption** means that all simple events have the same probability.

Consider the sample space $\mathcal{S} = \{x_1, x_2, \dots, x_N\}$ and the simple event $E_i = \{x_i\}$ such that

$$P(E_i) = P(\{x_i\}) = p \quad \text{for every } i.$$

- a. Find the value of p . $= \frac{1}{n}$
- b. Find $P(\{x_1, x_2, x_5\})$. $= \frac{3}{n}$
- c. What is $P(E)$ for any event E of $\mathcal{S}$?

$$P(E) = \frac{|E|}{n}$$

Theorem.

If we assume that each simple event in sample space S is as likely to occur as any other, then the probability of an arbitrary event A in S is given by

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{N(A)}{N}$$

Example 6. The following table summarizes the number of students enrolled in Probability and Statistics this semester. If a student is randomly selected from this batch, what is the probability that:

- the student is male? ^(one male)
- the student is third year?
- the student is not a in the fourth year?
- the student is a male and in his fifth year?

Gender	Year Level			Tot
	X Third	y Fourth	Z Fifth	
M Male	50	20	10	80
F Female	20	10	5	35
Total	70	30	15	115

$$a) p(M) = \frac{80}{115} = 0.695$$

$$b) p(X) = \frac{70}{115} = 0.608$$

$$c) p(y') = 1 - p(y) = 1 - \frac{30}{115} = 0.739$$

$$d) p(M \cap Z) = \frac{10}{115} = 0.0869$$

or
at least

and
both

(e) female or 3rd year.

$$\begin{aligned} P(F \cup X) &= P(F) + P(X) - P(F \cap X) \\ &= \frac{35}{115} + \frac{70}{115} - \frac{20}{115} = 0.739 \end{aligned}$$

Textbook

Study

Examples
2.16

Exercises

12,13,14,15,16,17,20,21,22,23

Exercise 12. Consider randomly selecting a student at a large university and let A be the event that the selected student has a Visa card and B be the analogous event for MasterCard. Suppose that $P(A) = 0.6$ and $P(B) = 0.4$.

- Could it be the case that $P(A \cap B) = .5$? Why or why not? [Hint: See Exercise 24.]
- From now on, suppose that $P(A \cap B) = 0.3$. What is the probability that the selected student has at least one of these two types of cards?
- What is the probability that the selected student has neither type of card?
- Describe, in terms of A and B , the event that the selected student has a Visa card but not a MasterCard, and then calculate the probability of this event.
- Calculate the probability that the selected student has exactly one of the two types of cards.

* visa card $\rightarrow A$

* analogous
(master) $\rightarrow B$

$$P(A) = 0.6 \quad , \quad P(B) = 0.4$$

Ⓐ $P(A \cap B) = 0.5 > P(B) = 0.4 \rightarrow$ No it cannot

$$\textcircled{b} P(A \cap B) = 0.3$$

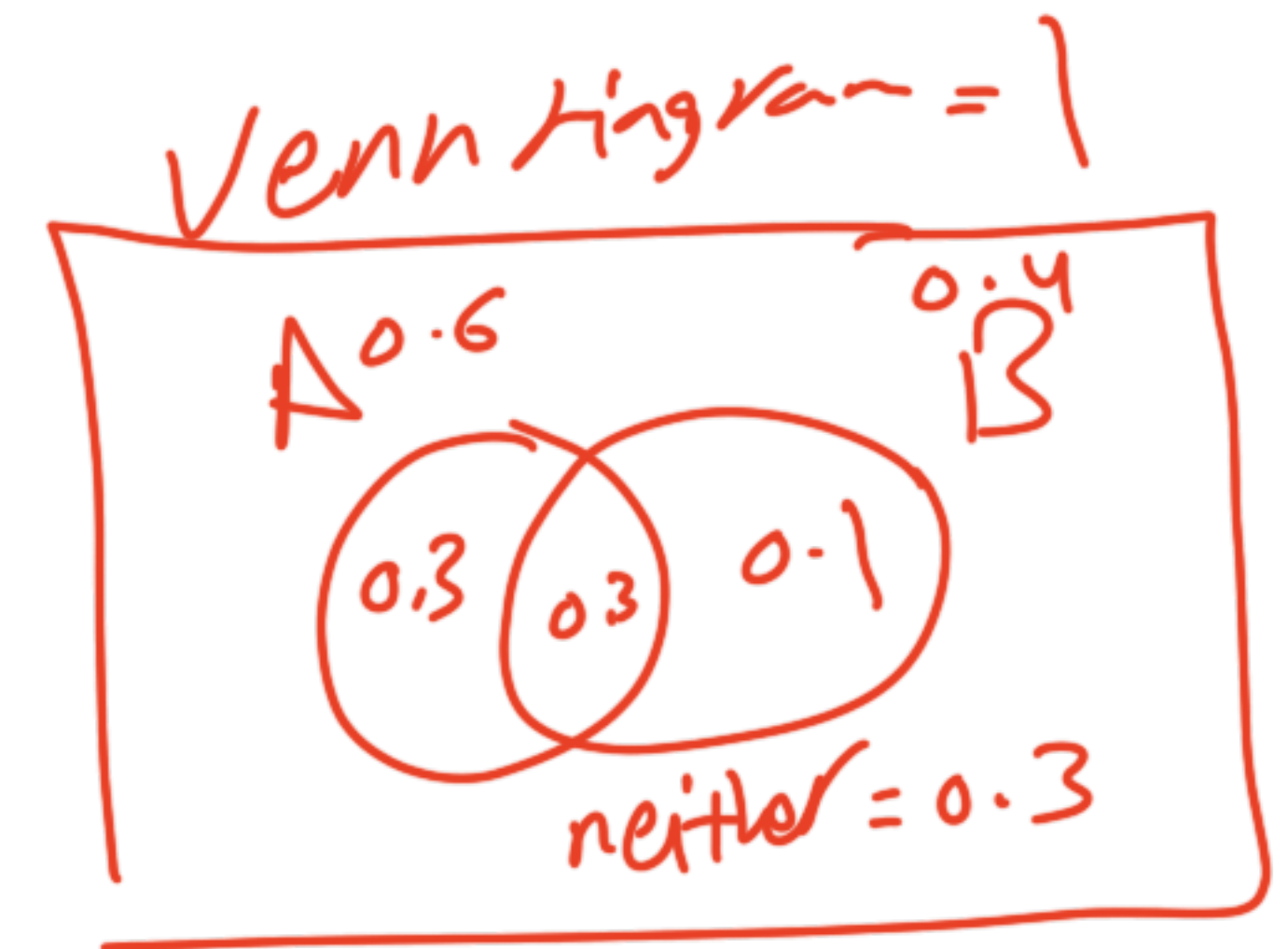
$$\begin{aligned} * P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.6 + 0.4 - 0.3 = 0.7 \end{aligned}$$

$$\begin{aligned} \text{---} \textcircled{c} P(A \cup B)' &= 1 - P(A \cup B) \\ &= 1 - 0.7 = 0.3 \end{aligned}$$

$$\textcircled{2} P(A) - P(A \cap B)$$

$$= 0.6 - 0.3 = 0.3$$

$$\begin{aligned} \textcircled{e} P(\text{exactly}) &= P(A \cup B) - P(A \cap B) \\ &= 0.7 - 0.3 = 0.4 \end{aligned}$$



Exercise 13. A computer consulting firm presently has bids out on three projects. Let $A_i = \{\text{awarded project } i\}$, for $i = 1, 2, 3$, and suppose that $P(A_1) = 0.22$, $P(A_2) = 0.25$, $P(A_3) = 0.28$, $P(A_1 \cap A_2) = 0.11$, $P(A_1 \cap A_3) = 0.05$, $P(A_2 \cap A_3) = 0.07$, $P(A_1 \cap A_2 \cap A_3) = 0.01$. Express in words each of the following events, and compute the probability of each event:

a. $A_1 \cup A_2$

b. $A'_1 \cap A'_2$ [*Hint:* $(A_1 \cup A_2)' = A'_1 \cap A'_2$]

c. $A_1 \cup A_2 \cup A_3$

d. $A'_1 \cap A'_2 \cap A'_3$

e. $A'_1 \cap A'_2 \cap A_3$

f. $(A'_1 \cap A'_2) \cup A_3$

Exercise 14. Suppose that 55% of all adults regularly consume coffee, 45% regularly consume carbonated soda, and 70% regularly consume at least one of these two products.

- What is the probability that a randomly selected adult regularly consumes both coffee and soda?
- What is the probability that a randomly selected adult doesn't regularly consume at least one of these two products?

$$p(c) = 0.55$$

$$p(s) = 0.45$$

$$(a) \quad p(c \cap s) = ?$$

$$(b) \quad p(c \cup s)' = 1 - p(c \cup s) \\ = 1 - 0.7 = 0.3$$

$$\underline{p(c \cup s) = 0.7}$$

$$p(c \cup s) = p(c) + p(s) - p(c \cap s) \\ 0.7 = 0.55 + 0.45 - p(c \cap s)$$

$$p(c \cap s) = 0.55 + 0.45 - 0.7 \\ = 0.3$$

Exercise 15. Consider the type of clothes dryer (gas or electric) purchased by each of five different customers at a certain store.

- a. If the probability that at most one of these purchases an electric dryer is .428, what is the probability that at least two purchase an electric dryer?
- b. If $P(\text{all five purchase gas}) = .116$ and $P(\text{all five purchase electric}) = .005$, what is the probability that at least one of each type is purchased?

Exercise 16. An individual is presented with three different glasses of cola, labeled C , D , and P . He is asked to taste all three and then list them in order of preference. Suppose the same cola has actually been put into all three glasses.

- a. What are the simple events in this ranking experiment, and what probability would you assign to each one?
- b. What is the probability that C is ranked first?
- c. What is the probability that C is ranked first and D is ranked last?

Exercise 17. Let A denote the event that the next request for assistance from a statistical software consultant relates to the SPSS package and let B be the event that the next request is for help with SAS. Suppose that $P(A) = 0.30$ and $P(B) = 0.50$.

- a. Why is it not the case that $P(A) + P(B) = 1$?
- b. Calculate $P(A')$.
- c. Calculate $P(A \cup B)$.
- d. Calculate $P(A' \cap B')$.

Exercise 20. A certain factory operates three different shifts. Over the last year, 200 accidents have occurred at the factory. Some of these can be attributed at least in part to unsafe working conditions, whereas the others are unrelated to working conditions. The accompanying table gives the percentage of accidents falling in each type of accident shift category.

Suppose one of the 200 accident reports is randomly selected from a file of reports, and the shift and type of accident are determined.

- a. What are the simple events?
- b. What is the probability that the selected accident was attributed to unsafe conditions?
- c. What is the probability that the selected accident did not occur on the day shift?

		Unsafe Conditions	Unrelated to Conditions
Shift	Day	10%	35%
	Swing	8%	20%
	Night	5%	22%

Exercise 21. An insurance company offers four different deductible levels—none, low, medium, and high—for its homeowner’s policyholders and three different levels—low, medium, and high—for its automobile policyholders. The accompanying table gives proportions for the various categories of policyholders who have both types of insurance.

Auto	Homeowner’s			
	N	L	M	H
L	.04	.06	.05	.03
M	.07	.10	.20	.10
H	.02	.03	.15	.15

For example, the proportion of individuals with both low homeowner’s deductible and low auto deductible is .06 (6% of all such individuals). Suppose an individual having both types of policies is randomly selected.

- a. What is the probability that the individual has a medium auto deductible and a high homeowner’s deductible?
- b. What is the probability that the individual has a low auto deductible? A low homeowner’s deductible?
- c. What is the probability that the individual is in the same category for both auto and homeowner’s deductibles?
- d. Based on your answer in part (c), what is the probability that the two categories are different?
- e. What is the probability that the individual has at least one low deductible level?
- f. Using the answer in part (e), what is the probability that neither deductible level is low?

Exercise 22. The route used by a certain motorist in commuting to work contains two intersections with traffic signals. The probability that he must stop at the first signal is .4, the analogous probability for the second signal is .5, and the probability that he must stop at at least one of the two signals is .7. What is the probability that he must stop

- a. At both signals?
- b. At the first signal but not at the second one?
- c. At exactly one signal?

Exercise 22. The computers of six faculty members in a certain department are to be replaced. Two of the faculty members have selected laptop machines and the other four have chosen desktop machines. Suppose that only two of the setups can be done on a particular day, and the two computers to be set up are randomly selected from the six (implying 15 equally likely outcomes; if the computers are numbered 1, 2, ..., 6, then one outcome consists of computers 1 and 2, another consists of computers 1 and 3, and so on).

- a. What is the probability that both selected setups are for laptop computers?
- b. What is the probability that both selected setups are desktop machines?
- c. What is the probability that at least one selected setup is for a desktop computer?
- d. What is the probability that at least one computer of each type is chosen for setup?

Question (ME 461).

A certain system can experience three different types of defects. Let A_1 , A_2 , and A_3 denote the events that the system has defects of type1, type2, and type3 respectively. Suppose that:

$P(A_1) = 0.12$, $P(A_2) = 0.07$, $P(A_3) = 0.05$, $P(A_1 \cup A_2) = 0.13$, $P(A_1 \cup A_3) = 0.14$, $P(A_2 \cup A_3) = 0.10$ and $P(A_1 \cap A_2 \cap A_3) = 0.01$.

- Compute $P(A'_1)$.
- Calculate $P(A_1 \cap A_2)$.
- Calculate $P(A_1 \cup A_2 \cup A_3)'$.

