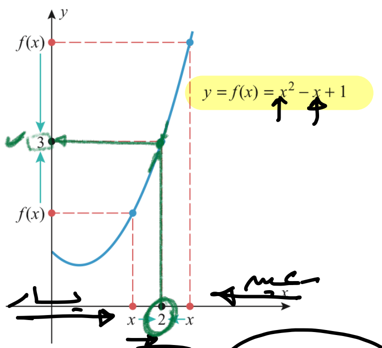
1.1 LIMITS (AN INTUITIVE APPROACH)

$$\lim_{x \to 2} (x^2 - x + 1) = 3 \tag{5}$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001	3	3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

يسار يمين

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

قيمة x قيمة y

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

ONE-SIDED LIMITS

The limit in (6) is called a ***two-sided limit*** because it requires the values of $f(x)$ to get closer and closer to L as values of x are taken from *either* side of $x = a$. However, some functions exhibit different behaviors on the two sides of an x-value a, in which case it is necessary to distinguish whether values of x near a are on the left side or on the right side of a for purposes of investigating limiting behavior. For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{12}$$

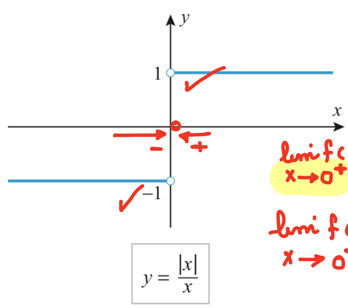
which is graphed in Figure 1.1.12. As x approaches 0 from the *right*, the values of $f(x)$ approach a limit of 1 [in fact, the values of $f(x)$ are exactly 1 for all such x], and similarly, as x approaches 0 from the *left*, the values of $f(x)$ approach a limit of -1. We denote these limits by writing

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1 \tag{13}$$

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

This leads to the general idea of a ***one-sided limit***.

$x \to$ ✓



$\lim_{x \to 0^+} f(x) = +1$ ✓

$\lim_{x \to 0^-} f(x) = -1$ ✓

> **1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW)** If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write
>
> $$\lim_{x \to a^+} f(x) = L \tag{14}$$
>
> and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write
>
> $$\lim_{x \to a^-} f(x) = L \tag{15}$$
>
> Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

THE RELATIONSHIP BETWEEN ONE-SIDED LIMITS AND TWO-SIDED LIMITS

In general, there is no guarantee that a function f will have a two-sided limit at a given point a; that is, the values of $f(x)$ may not get closer and closer to any *single* real number L as $x \to a$. In this case we say that

$$\lim_{x \to a} f(x) \quad \textbf{\textit{does not exist}}$$

Similarly, the values of $f(x)$ may not get closer and closer to a single real number L as $x \to a^+$ or as $x \to a^-$. In these cases we say that

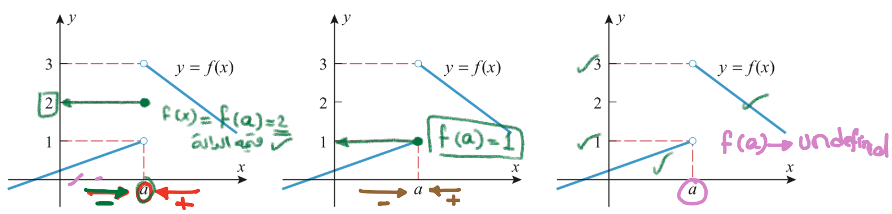
$$\lim_{x \to a^+} f(x) \quad \textbf{\textit{does not exist}}$$

or that

$$\lim_{x \to a^-} f(x) \quad \textbf{\textit{does not exist}}$$

In order for the two-sided limit of a function $f(x)$ to exist at a point a, the values of $f(x)$ must approach some real number L as x approaches a, and this number must be the same regardless of whether x approaches a from the left or the right. This suggests the following result, which we state without formal proof.

> **1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS** The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,
>
> $$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$



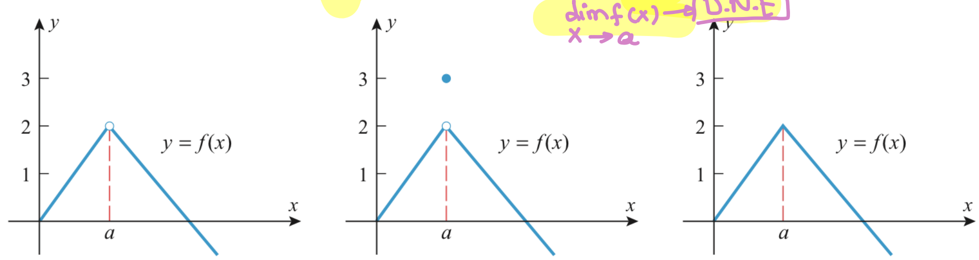
▶ **Figure 1.1.13**

$\lim_{x \to a^+} f(x) = 3$ ≠ $\lim_{x \to a} f(x)$ D.N.E

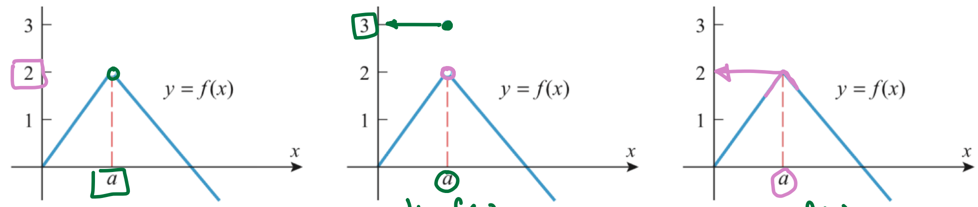
✓ $\lim_{x \to a^-} f(x) = 1$

$\lim_{x \to a^+} f(x) = 3$ ≠ $\lim_{x \to a^-} f(x) = 1$

$\lim_{x \to a} f(x) \to$ $\boxed{\text{D.N.E}}$



▲ **Figure 1.1.14**



▲ **Figure 1.1.14**

✓ $\lim_{x \to a^+} f(x) = 2$

✓ $\lim_{x \to a^-} f(x) = 2$

$\boxed{\lim_{x \to a} f(x) = 2}$

$f(a)$ undefined

$\lim_{x \to a^+} f(x) = 2$

$\lim_{x \to a^-} f(x) = 2$

$\lim_{x \to a} f(x) = 2$

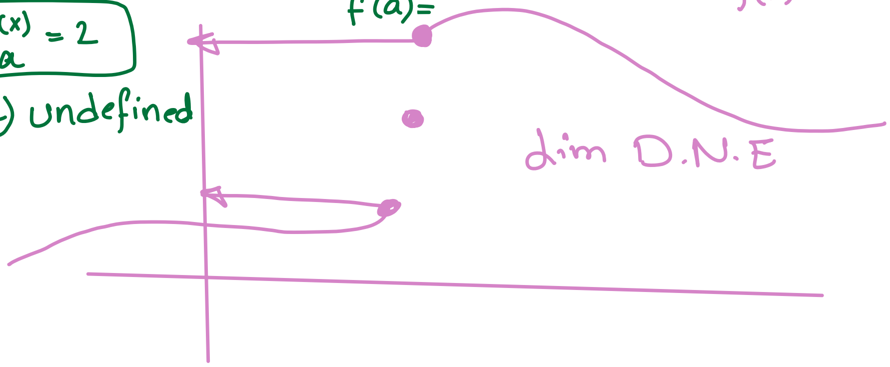
$f(a) =$

$\lim_{x \to a^+} f(x) = 2$

$\lim_{x \to a^-} f(x) = 2$

$\lim_{x \to a} f(x) = 2$

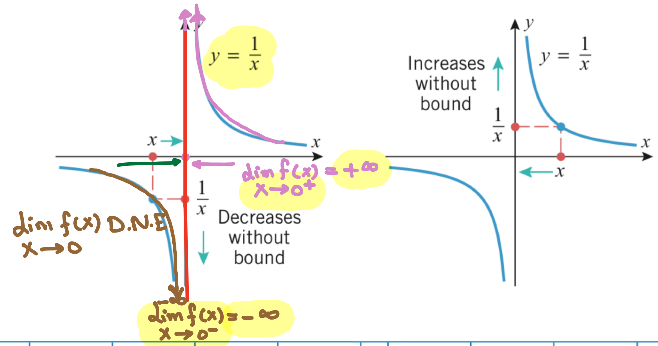
$f(a) = 2$



INFINITE LIMITS

Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound. For example, consider the behavior of $f(x) = 1/x$ for values of x near 0. It is evident from the table and graph in Figure 1.1.15 that as x-values are taken closer and closer to 0 from the right, the values of $f(x) = 1/x$ are positive and increase without bound; and as x-values are taken closer and closer to 0 from the left, the values of $f(x) = 1/x$ are negative and decrease without bound. We describe these limiting behaviors by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$



x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

Left side → ← Right side

▲ **Figure 1.1.15**

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

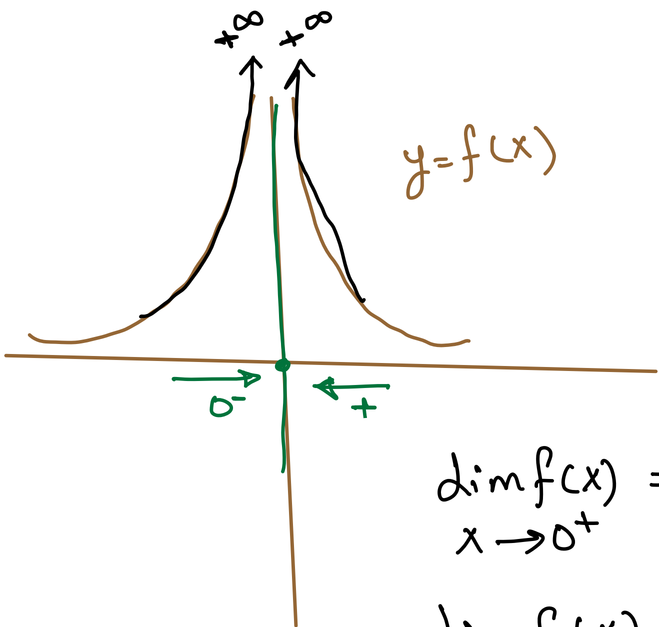
$$\lim_{x \to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$



$$\left.\begin{aligned}\lim_{x \to 0^+} f(x) &= +\infty \\ \lim_{x \to 0^-} f(x) &= +\infty\end{aligned}\right\}$$

$$\boxed{\lim_{x \to 0} f(x) = +\infty}$$

$$\left.\begin{aligned}\lim_{x \to a^+} f(x) &= \checkmark \\ \lim_{x \to a^-} f(x) &= \checkmark\end{aligned}\right\} = \longrightarrow \lim_{x \to a} f(x) = \checkmark$$

$$\neq \quad \lim f(x)$$

D.N.E