The photo shows comet Hale-Bopp as it passed the earth in 1997, due to return in 4380. One of the brightest comets of the past century, Hale-Bopp could be observed in the night sky by the naked eye for about 18 months. It was named after its discoverers Alan Hale and Thomas Bopp, who first obse ved it by telescope in 1995 (Hale in New Mexico and Bopp in Arizona). In Section 10.6 you will see how polar coordinates provide a convenient equation for the elliptical path of the comet's orbit.

Jeff Schneiderman / Moment Open / Getty Images

10 Parametric Equations and Polar Coordinates

SO FAR WE HAVE DESCRIBED plane curves by giving y as a function of x $[y = f(x)]$ or x as a function of y $[x = g(y)]$ or by giving a relation between x and y that defines y implicitly as a function of x $[f(x, y) = 0]$. In this chapter we discuss two new methods for describing curves.

Some curves, such as the cycloid, are best handled when both x and y are given in terms of a third variable t called a parameter $[x = f(t), y = g(t)]$. Other curves, such as the cardioid, have their most convenient description when we use a new coordinate system, called the polar coordinate system.

Calculus 3

10.1 Curves Defined by Parametric Equations

Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time t and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve.

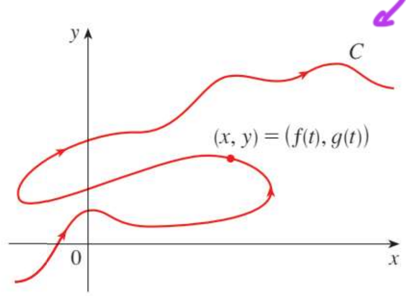


FIGURE 1

Parametric Equations

Suppose that x and y are both given as functions of a third variable t, called a **parameter**, by the equations

$$\boxed{x = f(t)} \qquad \boxed{y = g(t)}$$

which are called **parametric equations**. Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve called a **parametric curve**. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and in this case we can interpret $(x, y) = (f(t), g(t))$ as the position of a moving object at time t.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

SOLUTION Each value of t gives a point on the curve, as shown in the table. For instance, if $t = 1$, then $x = -1$, $y = 2$ and so the corresponding point is $(-1, 2)$. In Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

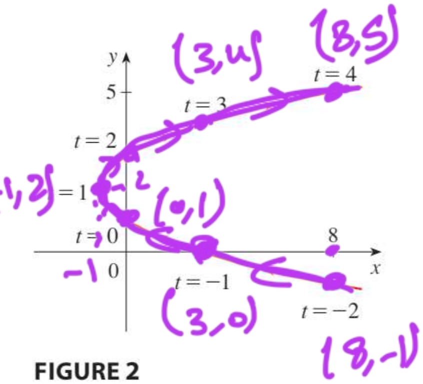
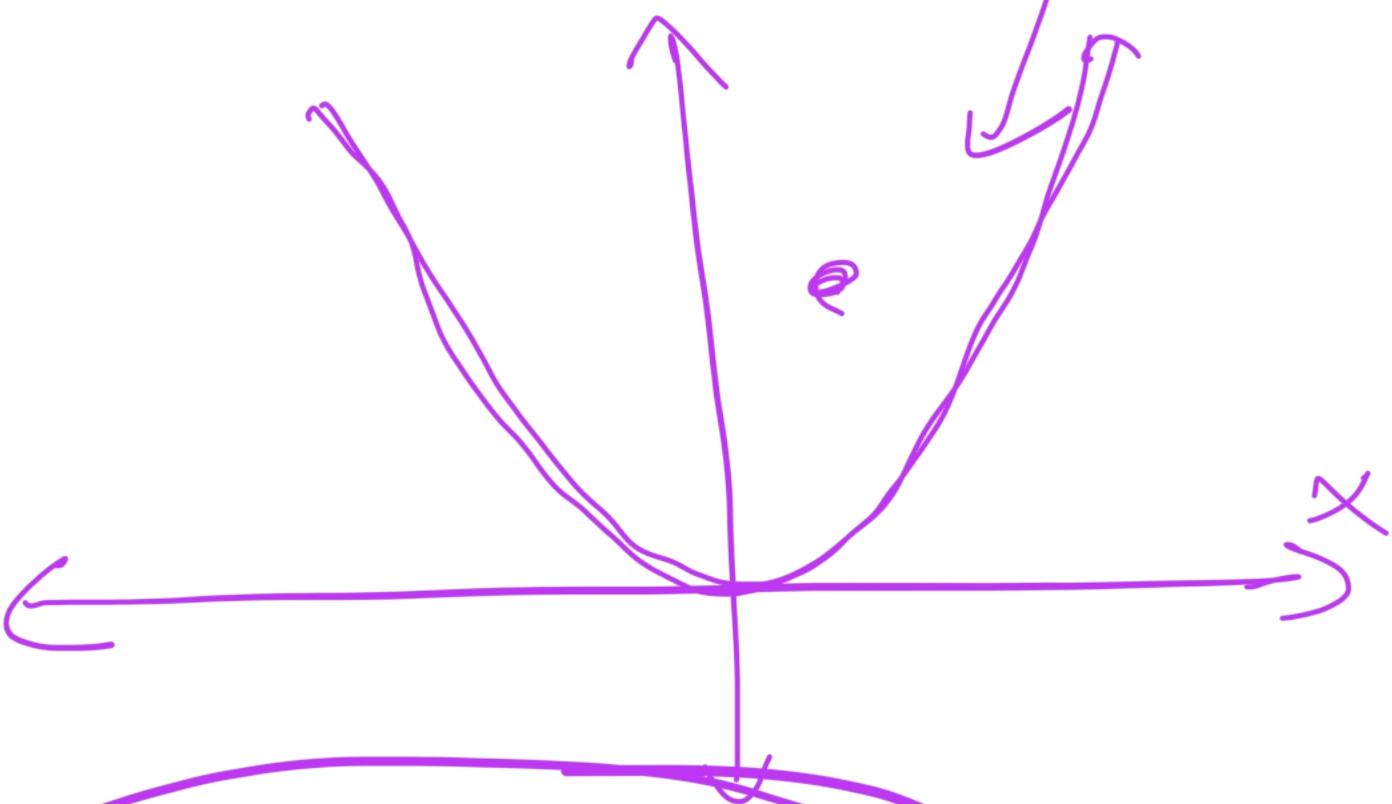


FIGURE 2

$\boxed{y = x^2}$

y a



parbolic

$\boxed{x = t}$ $\boxed{y = t^2}$

$(-3)^2$

t	x	y
-3	-3	9
-2	-2	4
-1	-1	1
0	0	0

Ex ②

$x = t^2 - 2t$ $\to (-2)^2 - 2(-2)$

$(-1)^2 - 2(-1)$

$1 + 2$

$4 + 4$

$y = t + 1$ $\to (-2) + 1$

$-1 + 1$ $-2 + 1$

t	x	y
-2	8	-1
-1	3	0
⋮	⋮	⋮

A particle whose position at time t is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. In fact, from the second equation we obtain $t = y - 1$ and substitution into the first equation gives

$$x = t^2 - 2t = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3$$

Since the equation $x = y^2 - 4y + 3$ is satisfied for all pairs of x- and y-values generated by the parametric equations, every point (x, y) on the parametric curve must lie on the parabola $x = y^2 - 4y + 3$ and so the parametric curve coincides with at least part of this parabola. Because t can be chosen to make y any real number, we know that the parametric curve is the entire parabola. ■

It is not always possible to eliminate the parameter from parametric equations. There are many parametric curves that don't have an equivalent representation as an equation in x and y.

In Example 1 we found a Cartesian equation in x and y whose graph coincided with the curve represented by parametric equations. This process is called **eliminating the parameter**; it can be helpful in identifying the shape of the parametric curve, but we lose some information in the process. The equation in x and y describes the curve the particle travels along, whereas the parametric equations have additional advantages—they tell us *where* the particle is at any given *time* and indicate the *direction* of motion. If you think of the graph of an equation in x and y as a road, then the parametric equations could track the motion of a car traveling along the road.

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number (including negative numbers). But sometimes we restrict t to lie in a particular interval. For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \leqslant t \leqslant 4$$

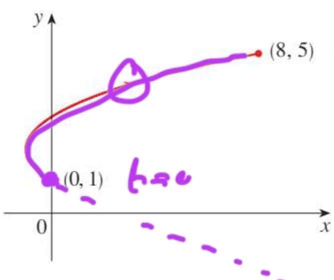


FIGURE 3

shown in Figure 3 is the part of the parabola in Example 1 that starts at the point $(0, 1)$ and ends at the point $(8, 5)$. The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.

In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leqslant t \leqslant b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

EXAMPLE 2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

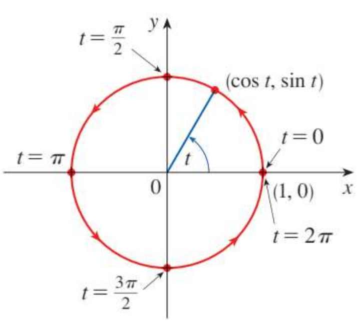


FIGURE 4

SOLUTION If we plot points, it appears that the curve is a circle. We can confirm this by eliminating the parameter t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Because $x^2 + y^2 = 1$ is satisfied for all pairs of x- and y-values generated by the parametric equations, the point (x, y) moves along the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians) shown in Figure 4. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$. ■

$$x = t^2 - 2t$$

$$y = t + 1$$

$$\boxed{y = f(x)}$$

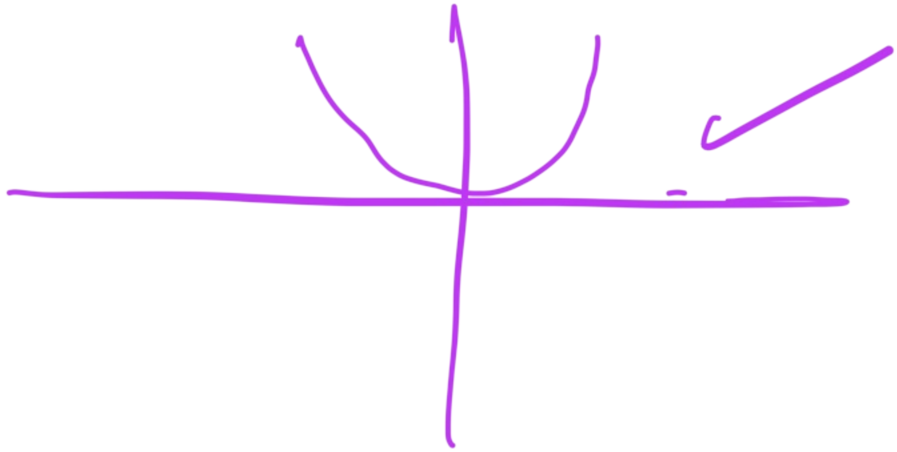
$$\boxed{t = y - 1}$$

$$x = (y-1)^2 - 2(y-1)$$

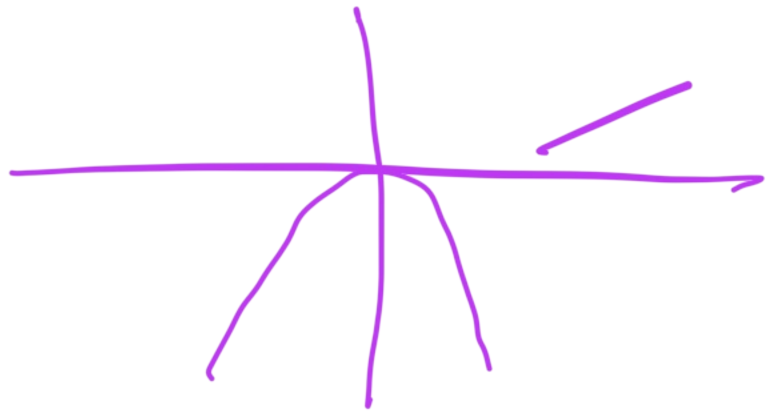
$$x = y^2 - 2y + 1 - 2y + 2$$

$$\boxed{x' = y^2 - 4y + 3}$$

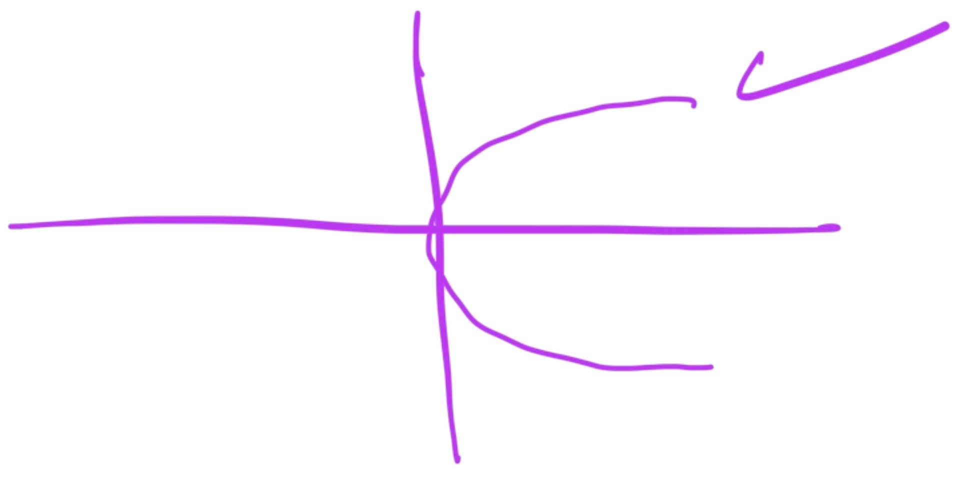
$y = x^2$



$y = -x^2$



$x = y^2$



$x = -y^2$

Parabola :

$x = t^2 - 2t$

$y = t + 1$

$0 \le t \le 4$

شرط

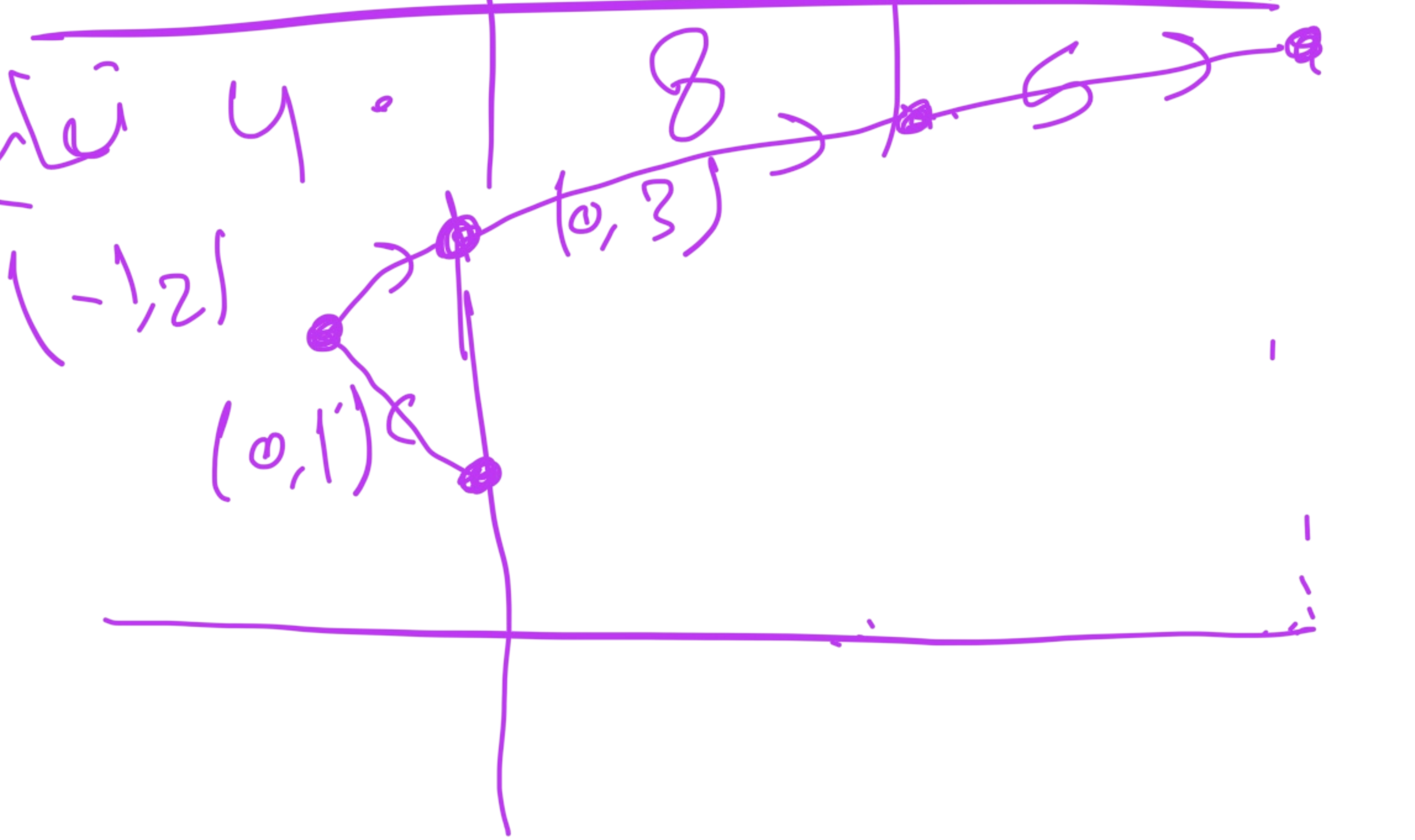
initial point

نقطة بداية

terminal point

نقطة النهاية

	t	x	y
بداية	0	0	1
	1	-1	2
	2	0	3
	3	3	4
نهاية	4	8	5



Ex (2)

الأشياء، المهمة

1. $\sin^2\boxed{(sh)\ fn^2} + \cos^2\boxed{(sh)\ fn^2} = 1 \to (1)$

2. $\sin 2t = 2\sin t\cos t$

3. $\cos 2t = \cos^2 t - \sin^2 t$

$= 2\cos^2 t - 1$

$= 1 - 2\sin^2 t$

$x = \cos t \qquad y = \sin t$

$0 \le t \le 2\pi$

$\sin^2 f + \cos^2 f = 1$

$$\boxed{\underline{y^2} + \underline{x^2} = 1}$$

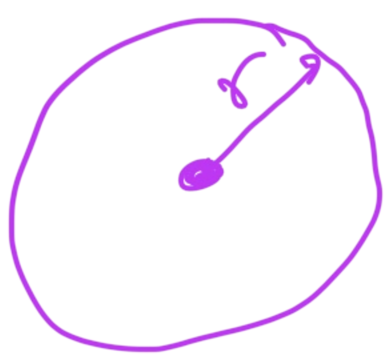
معادلة الدائرة

Circle Equation

$x^2 + y^2 = \sqrt{\underline{a}^2}$

↳ center = $(0,0)$
مركز

↳ radius = a
نصف قطر



$0 \le t \le 2\pi$

$x = \cos t$

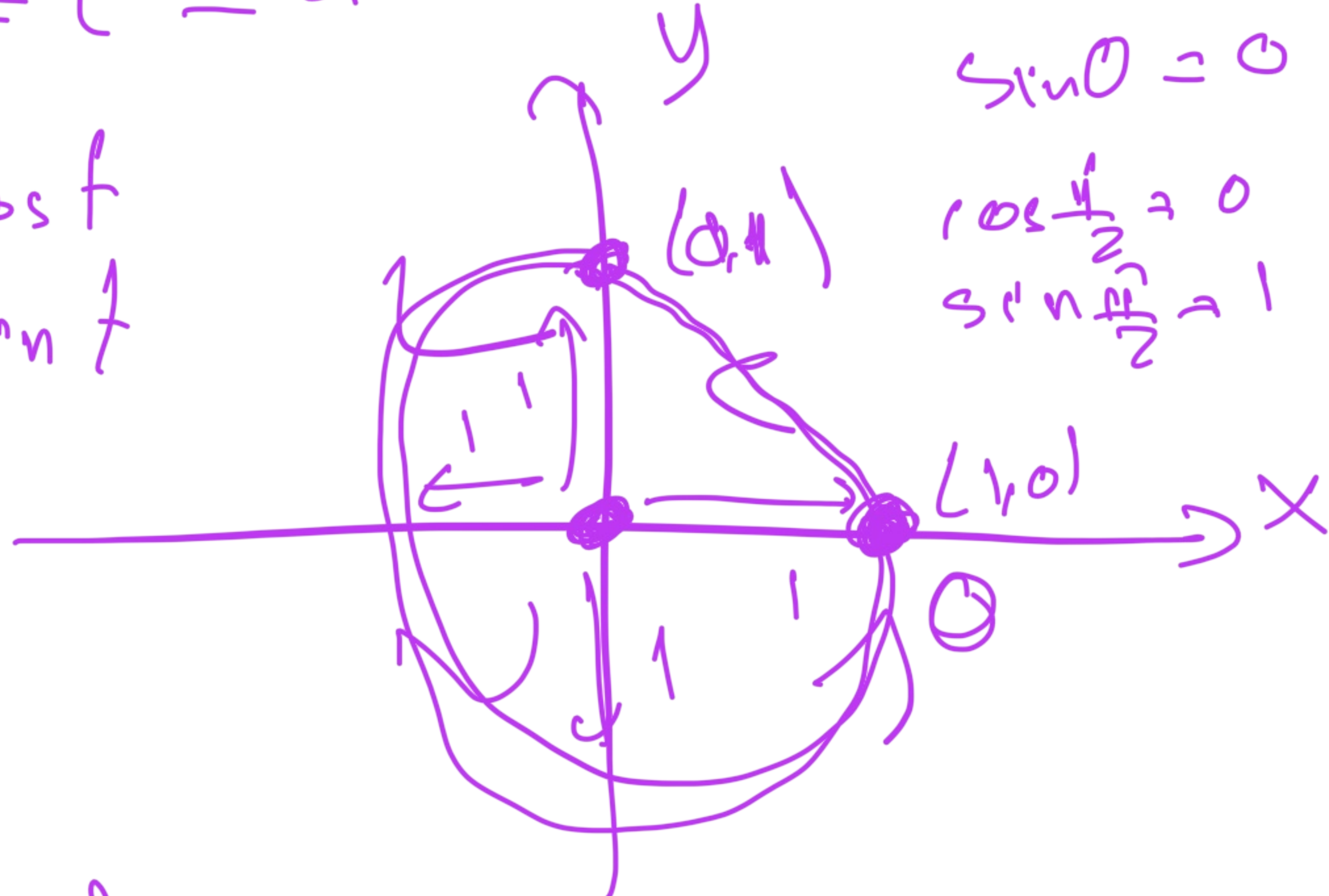
$y = \sin t$

$\cos 0 = 1$

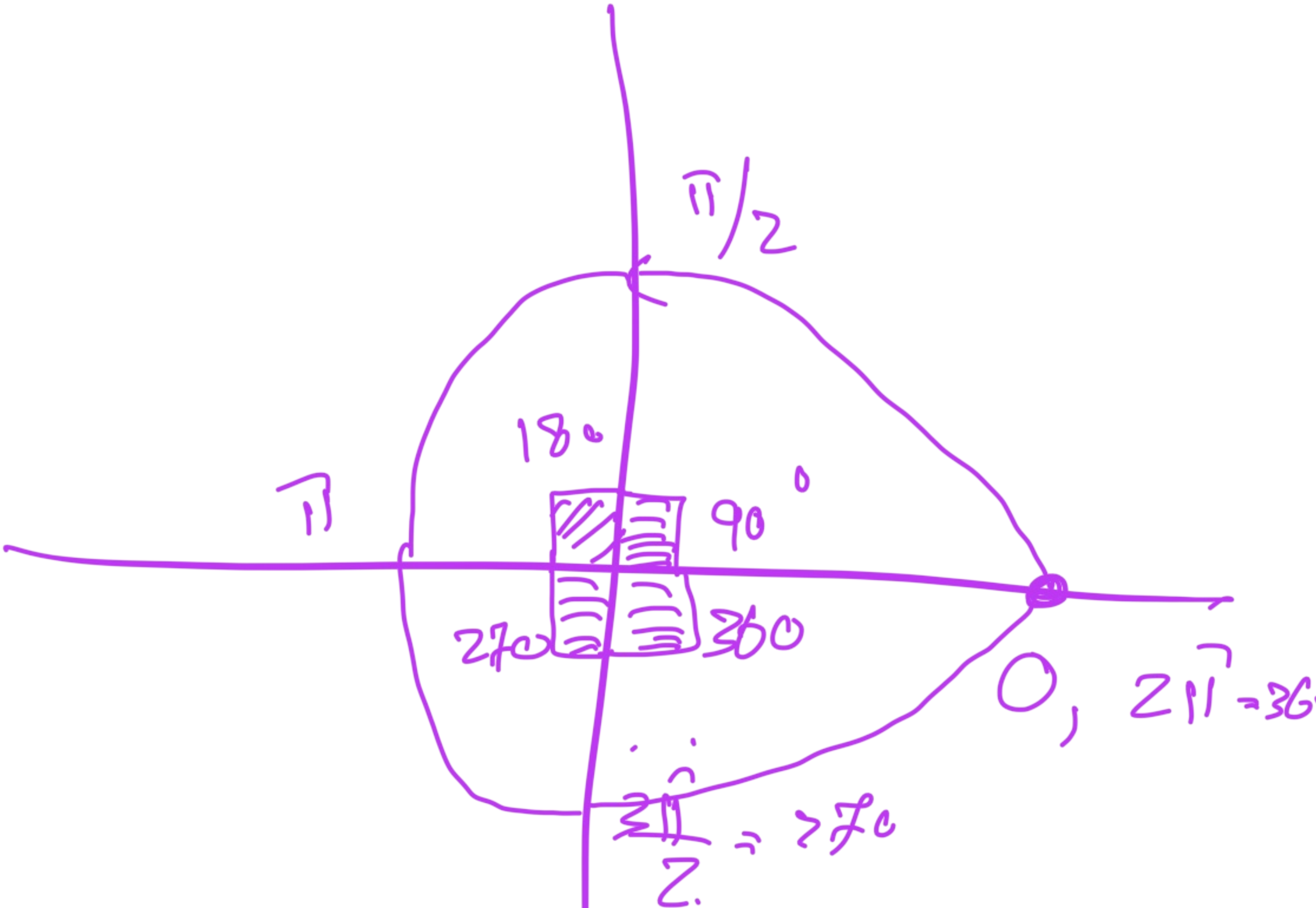
$\sin 0 = 0$

$\cos \frac{\pi}{2} = 0$

$\sin \frac{\pi}{2} = 1$



t	x	y
0	1	0
$90^\circ = \pi/2$	0	1



$$\boxed{\pi = 180^\circ}$$

clockwise مع عقارب الساعة

counter clockwise عكس عقارب الساعة

$t = 0$ $\quad$ $t \to \frac{\pi}{2}$ $\quad$ $x^2 + y^2 = 1$

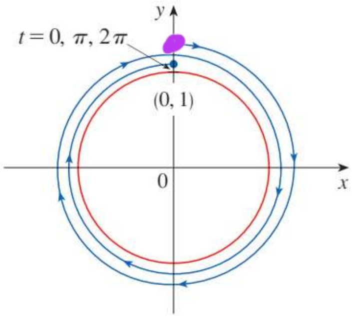


FIGURE 5

EXAMPLE 3 What curve is represented by the given parametric equations?

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

SOLUTION Again we have

$$x^2 + y^2 = \sin^2(2t) + \cos^2(2t) = 1$$

so the parametric equations again represent the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at $(0, 1)$ and moves *twice* around the circle in the clockwise direction as indicated in Figure 5. ■

EXAMPLE 4 Find parametric equations for the circle with center (h, k) and radius r.

SOLUTION One way is to take the parametric equations of the unit circle in Example 2 and multiply the expressions for x and y by r, giving $x = r\cos t$, $y = r\sin t$. You can verify that these equations represent a circle with radius r and center the origin, traced counterclockwise. We now shift h units in the x-direction and k units in the y-direction and obtain parametric equations of the circle (Figure 6) with center (h, k) and radius r:

$$x = h + r\cos t \qquad y = k + r\sin t \qquad 0 \leqslant t \leqslant 2\pi$$ ■

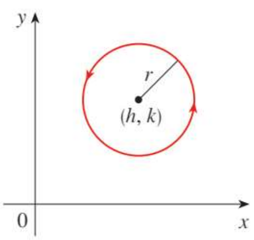


FIGURE 6
$x = h + r\cos t,\ y = k + r\sin t$

NOTE Examples 2 and 3 show that different parametric equations can represent the same curve. Thus we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced out in a particular way.

In the next example we use parametric equations to describe the motions of four different particles traveling along the same curve but in different ways.

EXAMPLE 5 Each of the following sets of parametric equations gives the position of a moving particle at time t.

(a) $x = t^3, \quad y = t$ $\qquad$ (b) $x = -t^3, \quad y = -t$ $\quad$ $t \to 1$

(c) $x = t^{3/2}, \quad y = \sqrt{t}$ $\quad$ $t \geq 0$ $\qquad$ (d) $x = e^{-3t}, \quad y = e^{-t}$

t	x	y
−1	−1	−1
0	0	0
1	1	1

In each case, eliminating the parameter gives $x = y^3$, so each particle moves along the cubic curve $x = y^3$; however, the particles move in different ways, as illustrated in Figure 7.

$t = 1 \to x = 0{,}04$, $y = 0{,}36$

(a) The particle moves from left to right as t increases.

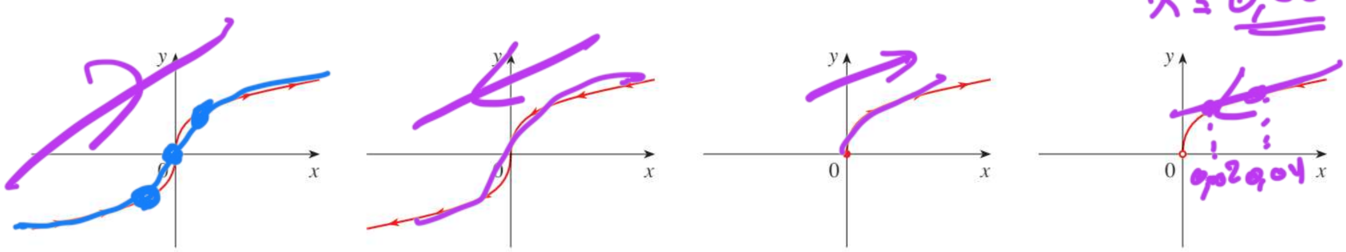
(b) The particle moves from right to left as t increases.

(c) The equations are defined only for $t \geqslant 0$. The particle starts at the origin (where $t = 0$) and moves to the right as t increases.

$t = 2$ $y = 0{,}13$

(d) Here $x > 0$ and $y > 0$ for all t. The particle moves from right to left and approaches the point (1,1) as t increases (through negative values) toward 0. As t further increases, the particle approaches, but does not reach, the origin.

$x = 0{,}002$



(a) $x = t^3, y = t$ $\qquad$ (b) $x = -t^3, y = -t$ $\qquad$ (c) $x = t^{3/2}, y = \sqrt{t}$ $\qquad$ (d) $x = e^{-3t}, y = e^{-t}$

FIGURE 7 ■

$x = \sin 2t$

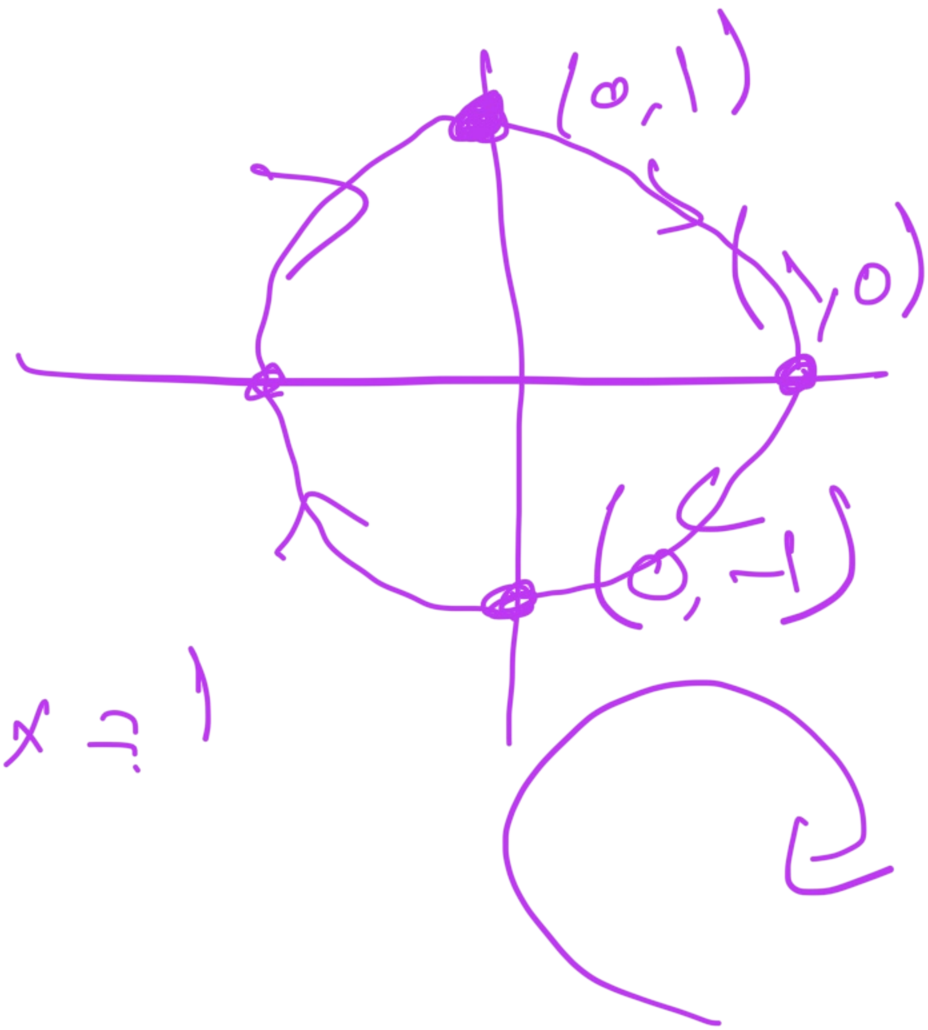
$y = \cos 2t$

$t = 0 \longrightarrow x = 0, \quad y = 1$

$t = \frac{\pi}{2} \longrightarrow x = 0, \quad y = -1$

$\sin\left(\cancel{2}\,\frac{\pi}{\cancel{2}}\right)$

$\cos\left(\cancel{2}\,\frac{\pi}{\cancel{2}}\right)$



$t = \frac{\pi}{4} \longrightarrow x = 1$

$\sin\left(\cancel{2}\,\frac{\pi}{\cancel{4}\,2}\right)$

$\cos\left(\cancel{2}\,\frac{\pi}{\cancel{4}\,2}\right) \longrightarrow y = 0$

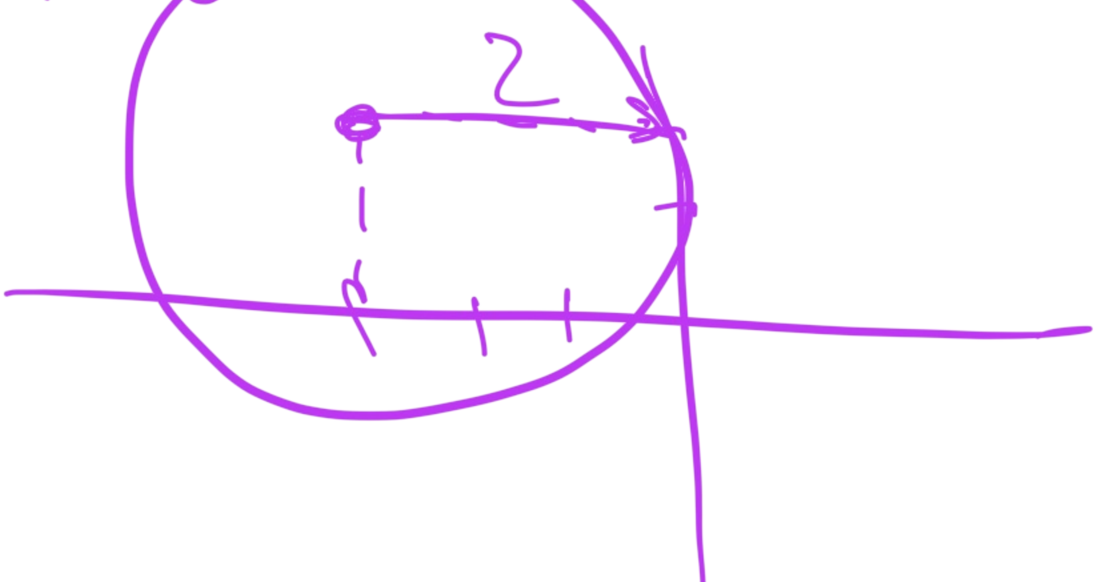
Circle equation

$\boxed{1}$ $(x-a)^2 + (y-b)^2 = r^2$

- center $= (a, b)$
- radius $= r$

Ex $(x+3)^2 + (y-2)^2 = \sqrt{4}$

- center $(-3, +2)$
- radius $= 2$



Parametric equation

$$\left.\begin{aligned} x &= h + r\cos t \\ y &= k + r\sin t \end{aligned}\right\} \rightarrow$$

نفس الرسم

دائرة ⇐

center = (h, k)

radius = r

Ex 5

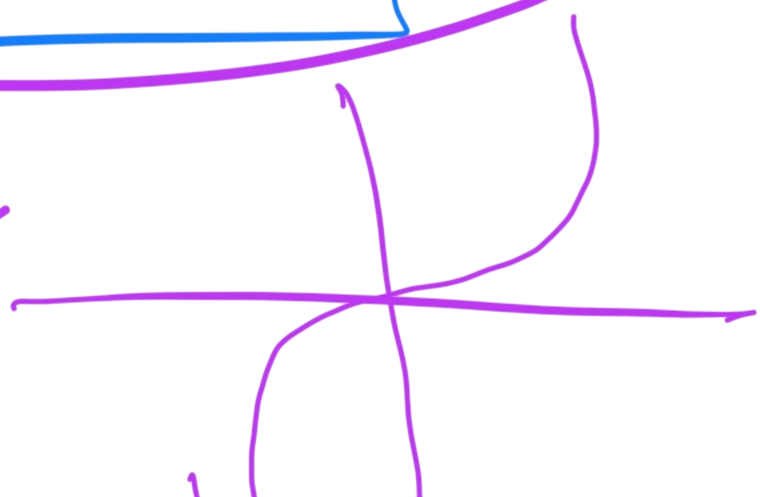
$x = t^3,$

$y = t$

$$\boxed{x = y^3}$$

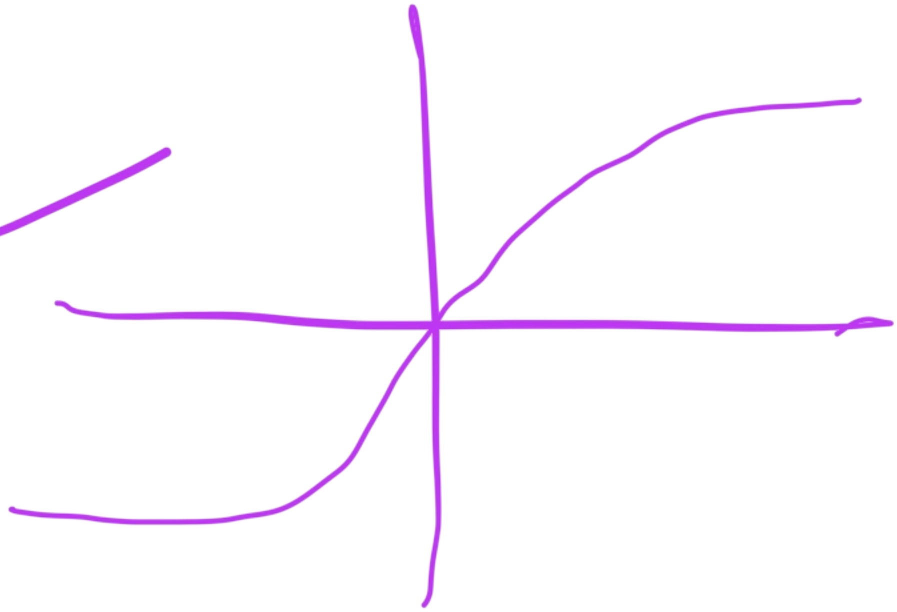
$$\boxed{\text{Cubic}}$$

$y = x^3$

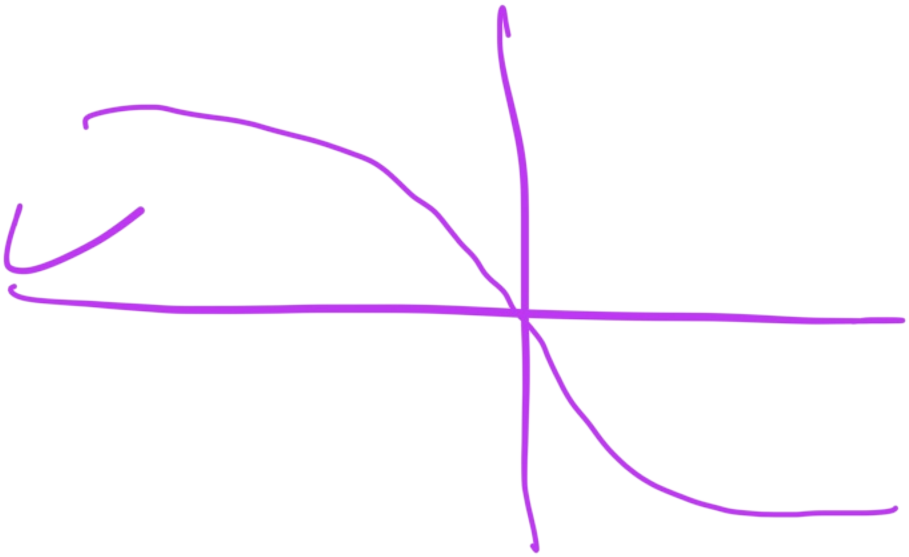


$y = -x^3$

$x = y^3$ ✓



$x = -y^3$ ✓



$\sqrt{-4}$ ✗

$\sqrt{6} \rightarrow 6 \geq 0$

$-1 \le \sin t \le 1$
$-1 \le x \le 1$
$y = x^2$

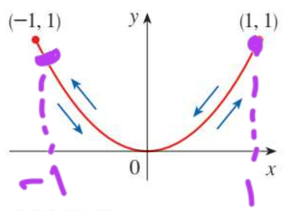
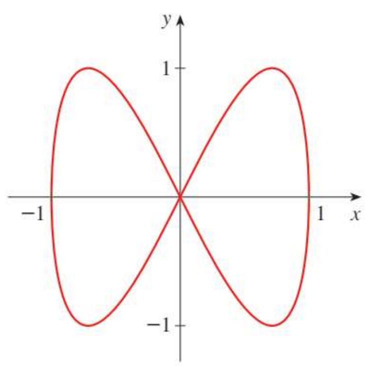


FIGURE 8

EXAMPLE 6 Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

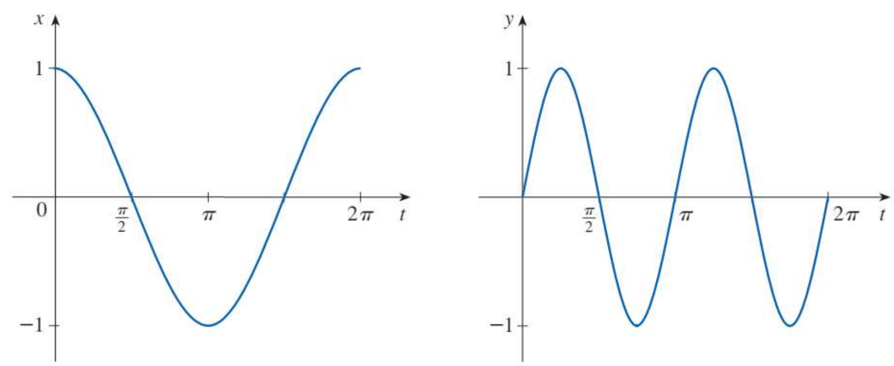
SOLUTION Observe that $y = (\sin t)^2 = x^2$ and so the point (x, y) moves on the parabola $y = x^2$. But note also that, since $-1 \leqslant \sin t \leqslant 1$, we have $-1 \leqslant x \leqslant 1$, so the parametric equations represent only the part of the parabola for which $-1 \leqslant x \leqslant 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1, 1)$ to $(1, 1)$. (See Figure 8.) ■

EXAMPLE 7 The curve represented by the parametric equations $x = \cos t$, $y = \sin 2t$ is shown in Figure 9. It is an example of a *Lissajous figure* (see Exercise 63). It is possible to eliminate the parameter, but the resulting equation $(y^2 = 4x^2 - 4x^4)$ isn't very helpful. Another way to visualize the curve is to first draw graphs of x and y individually as functions of t, as shown in Figure 10.



$x = \cos t$, $y = \sin 2t$

FIGURE 9



$x = \cos t$ $\qquad$ $y = \sin 2t$

FIGURE 10

We see that as t increases from 0 to $\pi/2$, x decreases from 1 to 0 while y starts at 0, increases to 1, and then returns to 0. Together these descriptions produce the portion of the parametric curve that we see in the first quadrant. If we proceed similarly, we get the complete curve. (See Exercises 31–33 for practice with this technique.) ■

Graphing Parametric Curves with Technology

Most graphing software applications and graphing calculators can graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

The next example shows that parametric equations can be used to produce the graph of a Cartesian equation where x is expressed as a function of y. (Some calculators, for instance, require y to be expressed as a function of x.)

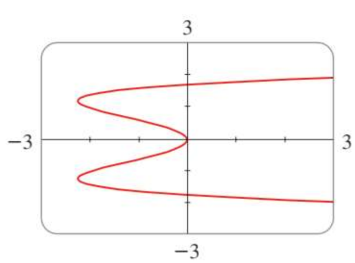


FIGURE 11

EXAMPLE 8 Use a calculator or computer to graph the curve $x = y^4 - 3y^2$.

SOLUTION If we let the parameter be $t = y$, then we have the equations

$$x = t^4 - 3t^2 \qquad y = t$$

Using these parametric equations to graph the curve, we obtain Figure 11. It would be possible to solve the given equation $(x = y^4 - 3y^2)$ for y as four functions of x and graph them individually, but the parametric equations provide a much easier method. ■

In general, to graph an equation of the form $x = g(y)$, we can use the parametric equations

$$x = g(t) \qquad y = t$$

In the same spirit, notice that curves with equations $y = f(x)$ (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

$$x = t \qquad y = f(t)$$

Graphing software is particularly useful for sketching complicated parametric curves. For instance, the curves shown in Figures 12, 13, and 14 would be virtually impossible to produce by hand.

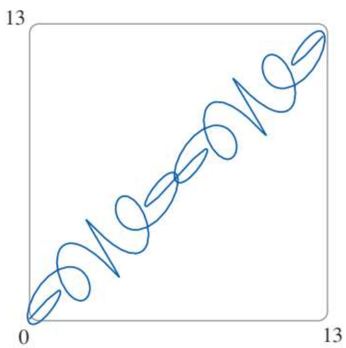


FIGURE 12
$x = t + \sin 5t$
$y = t + \sin 6t$

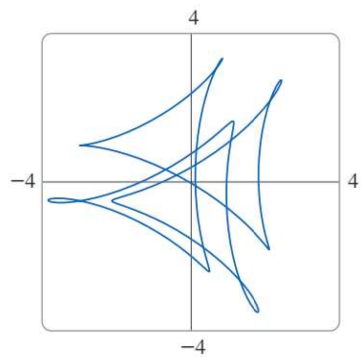


FIGURE 13
$x = \cos t + \cos 6t + 2 \sin 3t$
$y = \sin t + \sin 6t + 2 \cos 3t$

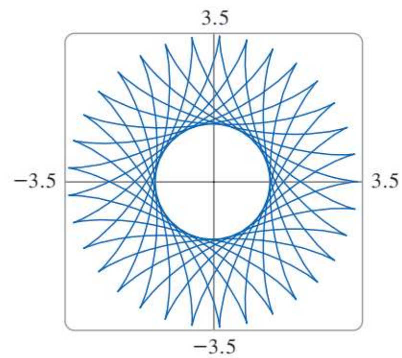


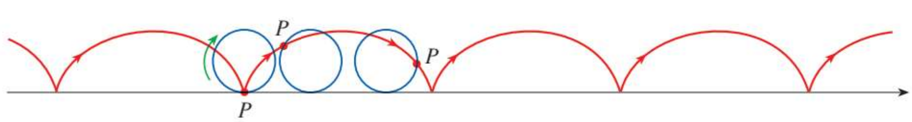
FIGURE 14
$x = 2.3 \cos 10t + \cos 23t$
$y = 2.3 \sin 10t - \sin 23t$

One of the most important uses of parametric curves is in computer-aided design (CAD). In the Discovery Project after Section 10.2 we will investigate special parametric curves, called **Bézier curves**, that are used extensively in manufacturing, especially in the automotive industry. These curves are also employed in specifying the shapes of letters and other symbols in PDF documents and laser printers.

The Cycloid

EXAMPLE 9 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid**. (Think of the path traced out by a pebble stuck in a car tire; see Figure 15.) If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

FIGURE 15



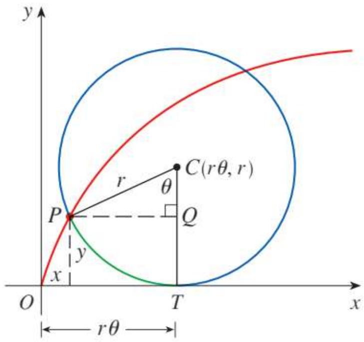


FIGURE 16

SOLUTION We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see from Figure 16 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 16 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

$$\boxed{1} \qquad x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \leqslant \theta \leqslant 2\pi$. Although Equations 1 were derived from Figure 16, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of θ (see Exercise 48).

Although it is possible to eliminate the parameter θ from Equations 1, the resulting Cartesian equation in x and y is very complicated [$x = r\cos^{-1}(1 - y/r) - \sqrt{2ry - y^2}$ gives just half of one arch] and not as convenient to work with as the parametric equations. ■

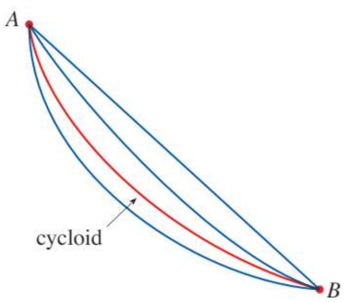


FIGURE 17

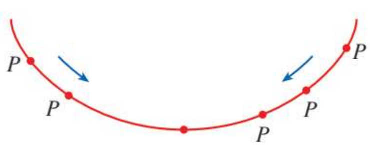


FIGURE 18

One of the first people to study the cycloid was Galileo; he proposed that bridges be built in the shape of cycloids and tried to find the area under one arch of a cycloid. Later this curve arose in connection with the **brachistochrone problem**: Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A. The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, as in Figure 17, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

The Dutch physicist Huygens had already shown by 1673 that the cycloid is also the solution to the **tautochrone problem**; that is, no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 18). Huygens proposed that pendulum clocks (which he invented) should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide arc or a small arc.

■ Families of Parametric Curves

EXAMPLE 10 Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a\tan t + \sin t$$

What do these curves have in common? How does the shape change as a increases?

SOLUTION We use a graphing calculator (or computer) to produce the graphs for the cases $a = -2, -1, -0.5, -0.2, 0, 0.5, 1$, and 2 shown in Figure 19. Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.

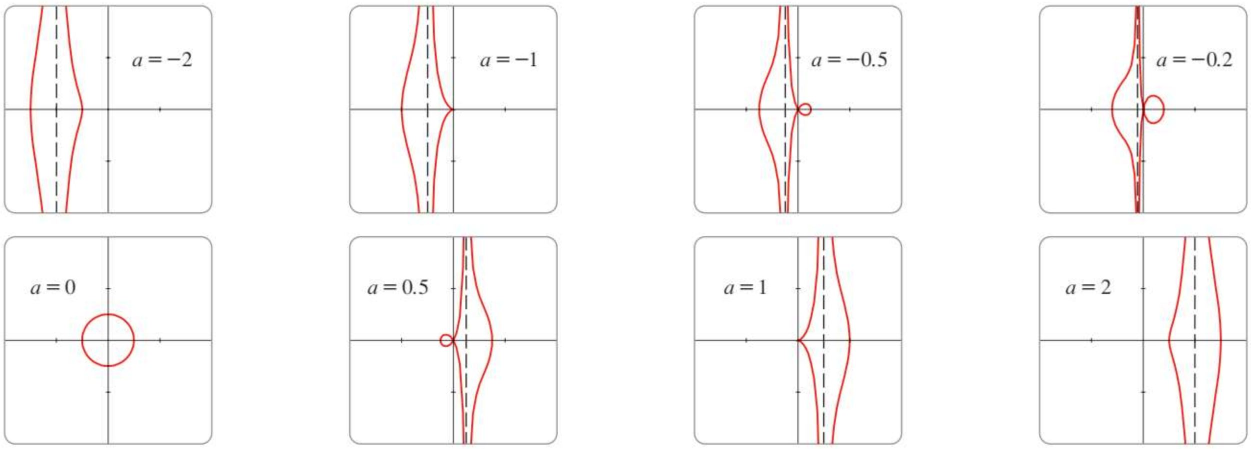


FIGURE 19
Members of the family $x = a + \cos t$, $y = a \tan t + \sin t$, all graphed in the viewing rectangle $[-4, 4]$ by $[-4, 4]$

When $a < -1$, both branches are smooth; but when a reaches -1, the right branch acquires a sharp point, called a *cusp*. For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When $a = 0$, both branches come together and form a circle (see Example 2). For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when $a = 1$. For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell. ■

10.1 Exercises

1–2 For the given parametric equations, find the points (x, y) corresponding to the parameter values $t = -2, -1, 0, 1, 2$.

1. $x = t^2 + t, \quad y = 3^{t+1}$

2. $x = \ln(t^2 + 1), \quad y = t/(t + 4)$

3–6 Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

3. $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \leqslant t \leqslant 2$

4. $x = t^3 + t, \quad y = t^2 + 2, \quad -2 \leqslant t \leqslant 2$

5. $x = 2^t - t, \quad y = 2^{-t} + t, \quad -3 \leqslant t \leqslant 3$

6. $x = \cos^2 t, \quad y = 1 + \cos t, \quad 0 \leqslant t \leqslant \pi$

7–12
(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

7. $x = 2t - 1, \quad y = \frac{1}{2}t + 1$

8. $x = 3t + 2, \quad y = 2t + 3$

9. $x = t^2 - 3, \quad y = t + 2, \quad -3 \leqslant t \leqslant 3$

10. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \leqslant t \leqslant 2\pi$

11. $x = \sqrt{t}, \quad y = 1 - t$

12. $x = t^2, \quad y = t^3$

13–22
(a) Eliminate the parameter to find a Cartesian equation of the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

13. $x = 3 \cos t, \quad y = 3 \sin t, \quad 0 \leqslant t \leqslant \pi$

14. $x = \sin 4\theta, \quad y = \cos 4\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

15. $x = \cos \theta, \quad y = \sec^2\theta, \quad 0 \leqslant \theta < \pi/2$

16. $x = \csc t, \quad y = \cot t, \quad 0 < t < \pi$

17. $x = e^{-t}, \quad y = e^t$

18. $x = t + 2, \quad y = 1/t, \quad t > 0$

19. $x = \ln t, \quad y = \sqrt{t}, \quad t \geqslant 1$

20. $x = |t|, \quad y = |1 - |t||$

21. $x = \sin^2 t, \quad y = \cos^2 t$

22. $x = \sinh t, \quad y = \cosh t$

23–24 The position of an object in circular motion is modeled by the given parametric equations, where t is measured in seconds. How long does it take to complete one revolution? Is the motion clockwise or counterclockwise?

23. $x = 5 \cos t, \quad y = -5 \sin t$

24. $x = 3 \sin\left(\dfrac{\pi}{4}t\right), \quad y = 3 \cos\left(\dfrac{\pi}{4}t\right)$

25–28 Describe the motion of a particle with position (x, y) as t varies in the given interval.

25. $x = 5 + 2 \cos \pi t, \quad y = 3 + 2 \sin \pi t, \quad 1 \leqslant t \leqslant 2$

26. $x = 2 + \sin t, \quad y = 1 + 3 \cos t, \quad \pi/2 \leqslant t \leqslant 2\pi$

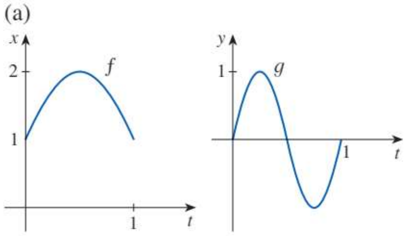
27. $x = 5 \sin t, \quad y = 2 \cos t, \quad -\pi \leqslant t \leqslant 5\pi$

28. $x = \sin t, \quad y = \cos^2 t, \quad -2\pi \leqslant t \leqslant 2\pi$

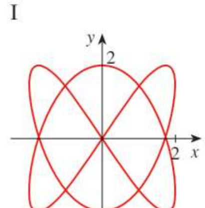
29. Suppose a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, where the range of f is $[1, 4]$ and the range of g is $[2, 3]$. What can you say about the curve?

30. Match each pair of graphs of equations $x = f(t)$, $y = g(t)$ in (a)–(d) with one of the parametric curves $x = f(t)$, $y = g(t)$ labeled I–IV. Give reasons for your choices.

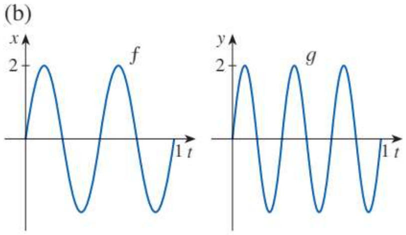
(a)



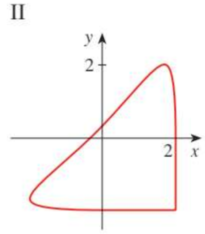
I



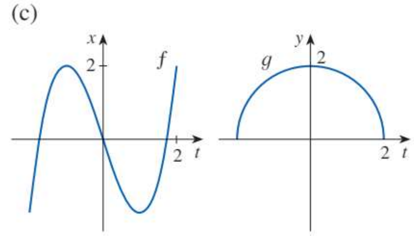
(b)



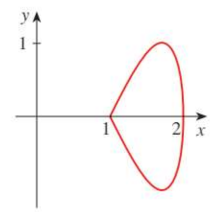
II



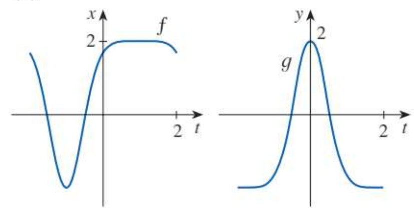
(c)



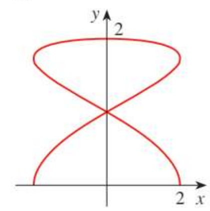
III



(d)

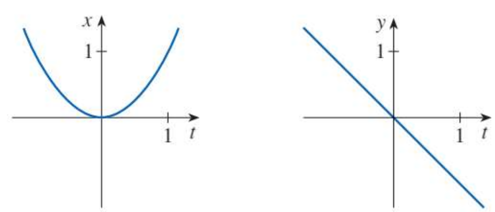


IV

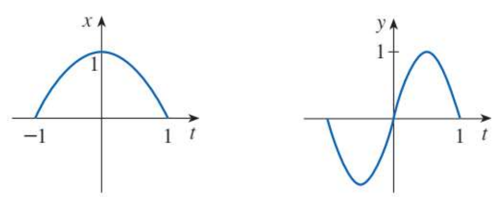


31–33 Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.

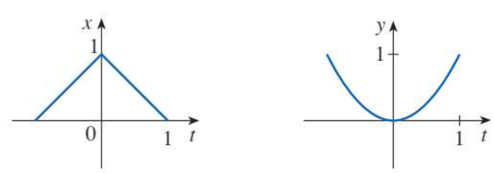
31.



32.



33.



34. Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices.

(a) $x = t^4 - t + 1, \quad y = t^2$

(b) $x = t^2 - 2t, \quad y = \sqrt{t}$

(c) $x = t^3 - 2t, \quad y = t^2 - t$

(d) $x = \cos 5t, \quad y = \sin 2t$

(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$