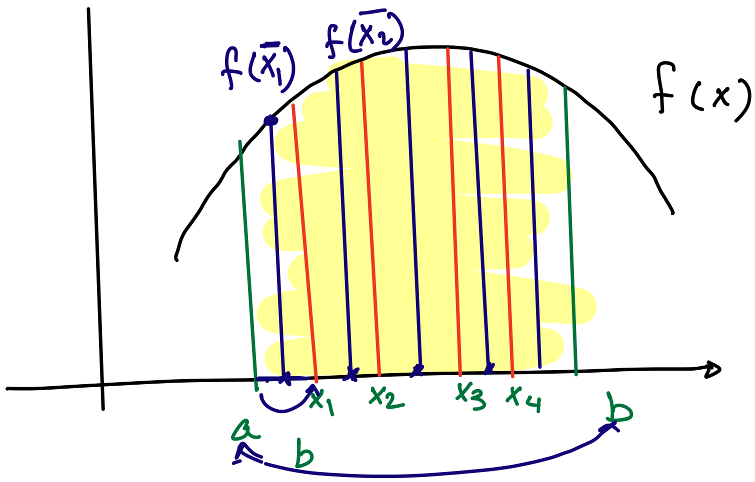



$n = \checkmark$

$= \int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx = \underline{M_n} = \Delta x\Big[\underbrace{f(\bar{x}_1)}_{f(a)} + f(\bar{x}_2) + f(\bar{x}_3) + \cdots + \underbrace{f(\bar{x}_n)}_{f(b)}$$

$$\Delta x = \frac{b-a}{n}$$

## 7.7 Approximate Integration

### The Midpoint and Trapezoidal Rules

In Section 5.2 we considered the case where $x_i^*$ in the Riemann sum is chosen to be the midpoint $\bar{x}_i$ of the subinterval $[x_{i-1}, x_i]$. Figure 2 shows the midpoint approximation $M_n$ for the area in Figure 1. It appears that $M_n$ is a better approximation than either $L_n$ or $R_n$.

**Midpoint Rule**

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $$\Delta x = \frac{b-a}{n}$$

and $$\bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

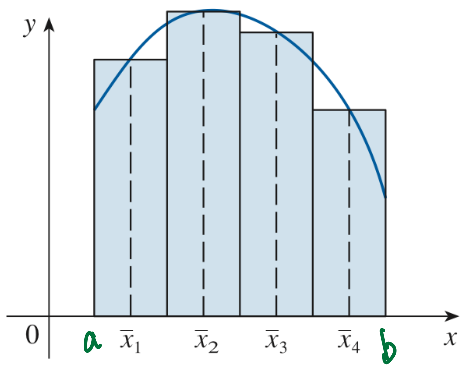



**FIGURE 2**
Midpoint approximation

Another approximation, called the Trapezoidal Rule,

**Trapezoidal Rule**

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

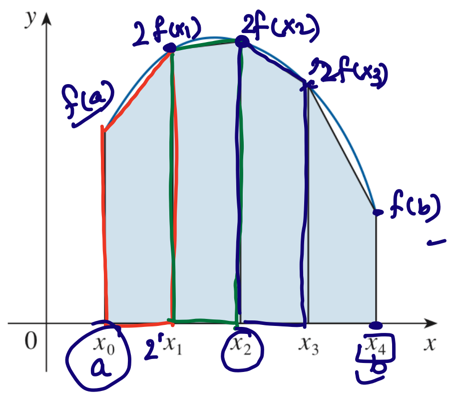



**FIGURE 3**
Trapezoidal approximation

$$\Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2 + \underbrace{f(x_n)}_{b}\right]$$

$a$

**EXAMPLE 1** Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 (1/x)\,dx$.

$a = 1 \qquad b = 2 \qquad n = 5$

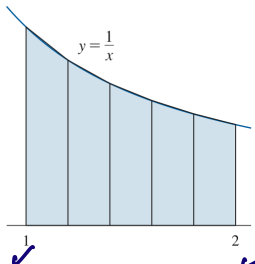


FIGURE 4

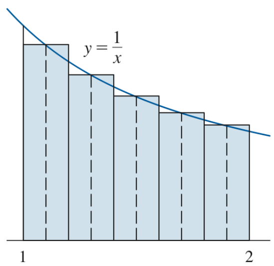


FIGURE 5

a] Trapezoidal rule.

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = 0.2$$

$$\boxed{\Delta x = 0.2}$$

1   1.2   1.4   1.6   1.8   2

$$f(x) = \frac{1}{x}$$

$$\int_a^b f(x)\,dx = \frac{\Delta x}{2}\left[f(a) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(b)\right]$$

$a = 1$, $b = 2$

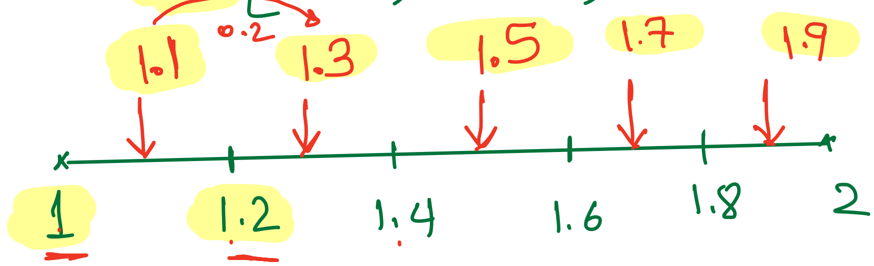
$$= \frac{0.2}{2}\left[\frac{1}{1} + 2\left(\frac{1}{1.2}\right) + 2\left(\frac{1}{1.4}\right) + 2\left(\frac{1}{1.6}\right) + 2\left(\frac{1}{1.8}\right) + \frac{1}{2}\right)$$

$$= 0.695635$$

$\boxed{2}$ Midpoint rule. $a = 1$ $b = 2$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = 0.2$$

$$\int_a^b f(x)\,dx = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \cdots f(\bar{x}_n)\right]$$




$$\int_1^2 \frac{1}{x}\,dx = 0.2\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$$

$$= 0.2\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right]$$

$$\simeq 0.691908$$

## Simpson's Rule

**Simpson**
Thomas Simpson was a weaver who taught himself mathematics and went on to become one of the best English mathematicians of the 18th century. What we call Simpson's Rule was actually known to Cavalieri and Gregory in the 17th century, but Simpson popularized it in his book *Mathematical Dissertations* (1743).

**Simpson's Rule**

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where $n$ is even and $\Delta x = (b - a)/n$.

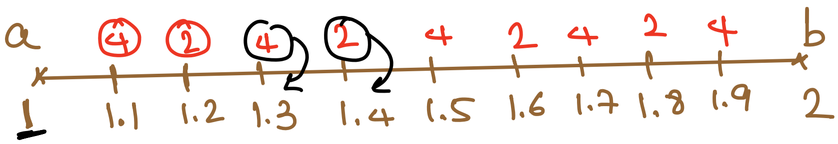
a   f(b)   2

**EXAMPLE 4** Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 (1/x)\,dx$.

4 2 4 2 4

$\boxed{1}$ $\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = 0.1$

$\boxed{2}$




$$\int_1^2 \frac{1}{x}\,dx = \frac{\Delta x}{3}\Big[f(a) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(b)\Big]$$

1   2

$$= \frac{0.1}{3}\Big[\frac{1}{1} + 4\left(\frac{1}{1.1}\right) + 2\left(\frac{1}{1.2}\right) + 4\left(\frac{1}{1.3}\right) + 2\left(\frac{1}{1.4}\right) + 4\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{1.6}\right) + 4\left(\frac{1}{1.7}\right) + 2\left(\frac{1}{1.8}\right) + 4\left(\frac{1}{1.9}\right) + \frac{1}{2}\Big]$$

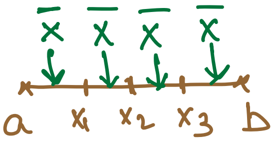
$$\approx 0.693150$$

$$\int_a^b f(x)\,dx$$

$\Delta x = \frac{b-a}{n} = \checkmark$

## Midpoint

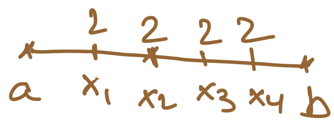
$M_n$




$$\Delta x\left[f(\bar{x}_1)+f(\bar{x}_2)+\cdots f(\bar{x}_n)\right]$$

## Trapezoidal

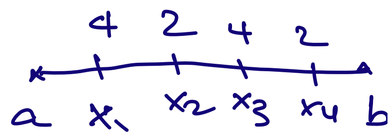
$T_n$




$$\frac{\Delta x}{2}\left[f(a)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+f(b)\right]$$

## Simpson

$S_n$




$$\frac{\Delta x}{3}\left[f(a)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+f(b)\right]$$