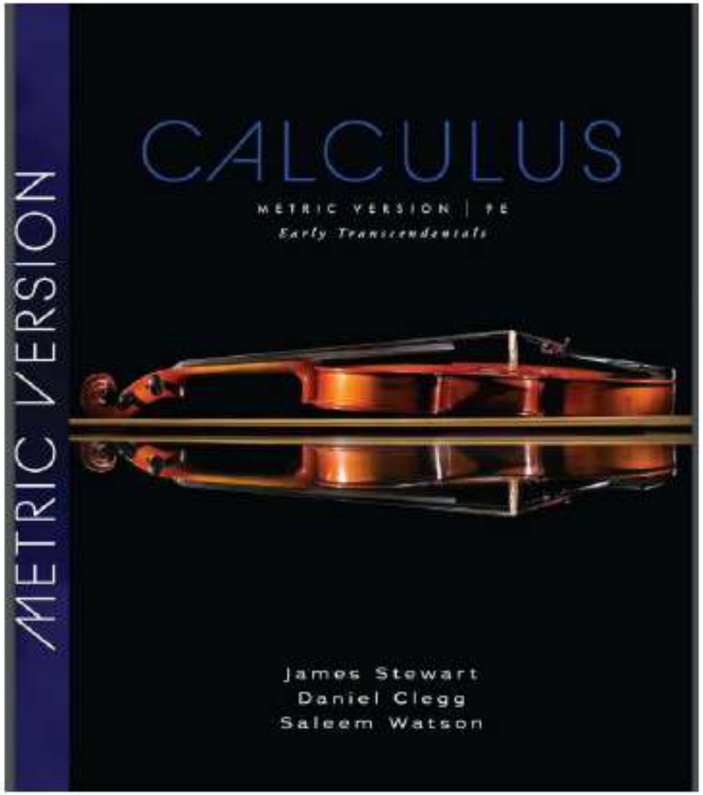
Calculus I



This helping matiral is taken from Stewart, J., Clegg, D.K., & Watson, S. (2021). Calculus: Early Transcendentals (9th ed.).

- Limits
- differentiation
- Integration

2 Limits and Derivatives

النهايات الاشتقاق

خط التماس السرعة اللحظية

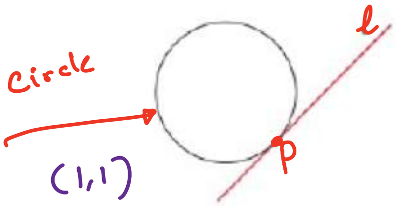
2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

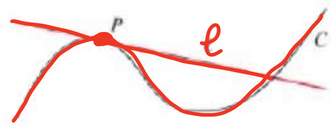
The Tangent Problem

تمس

The word *tangent* is derived from the Latin word *tangens*, which means "touching." We can think of a tangent to a curve as a line that touches the curve and follows the same direction as the curve at the point of contact. How can this idea be made precise?



(a)



(b)

$\boxed{y - y_1 = m(x - x_1)}$ ← نقطة + ميل ✓ → point slope form

(1,1)

نقطتين النقطة لبتقاطع مع محور السينات x xx

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

x y

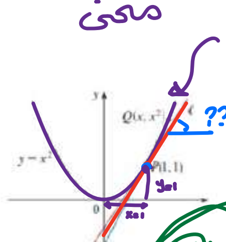
منحنى

نقطة واقعة على كل من الخط المماس و المنحنى

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

$$m_{tan} = \frac{y_2 - y_1}{x_2 - x_1}$$

← معادلة ميل اي خط مستقيم معرفة نقطتين واحدة على الخط



x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

1st point (1,1)

1 2 3 4

0 0.5 0.9 0.99 0.999 1 1.001 1.01 1.1 1.5

$x = 1.001 \rightarrow y = ??$

$y = x^2 = (1.001)^2 = (x_2, y_2)$

$y = x^2$ ← محددة منحنى

افرض نقطة ← (x_2, y_2)

$x_2 = x \longrightarrow y_2 = x^2 \longrightarrow$ point (x, x^2)

↑ النقطة الثانية

Point ②

$(x, x^2) \longrightarrow y = x^2$, $x_2 = x$, $y_2 = x^2$

Point ①

Given $(1, 1)$

نقطة التماس

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2 - 1}{x - 1}$$

الصورة العامة لميل الخط المستقيم الحامل

curve $y = x^2$

at any point

> 1

x افرض	$m_{PQ} = \frac{x^2-1}{x-1}$ ميل
2	3 → $x=2 \to \frac{2^2-1}{2-1} = 3$
1.5	2.5 → $x=1.5 \to m = \frac{1.5^2-1}{1.5-1} = 2.5$
1.1	2.1 → $x=1.1 \to m = \frac{1.1^2-1}{1.1-1} = 2.1$
1.01	2.01
1.001	2.001

< 1 $\quad m = \frac{x^2-1}{x-1}$

x	m_{PQ}
0	1 → $x=0 \to m = \frac{0^2-1}{0-1} = 1$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

$x = 1$

x	0	0.5	0.9	0.99	0.999	1.001	1.01	1.1	1.5	2
m	1	1.5	1.9	1.99	1.999	2.001	2.01	2.1	2.5	3

$\boxed{2}$

كلما اقتربت الـ x من $\boxed{1}$ (نقطة التماس) ← اقترب الميل من $\boxed{2}$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through $(1, 1)$ as

$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (1, 1) \to$ Given

$m \simeq 2$

$y - 1 = 2(x - 1)$

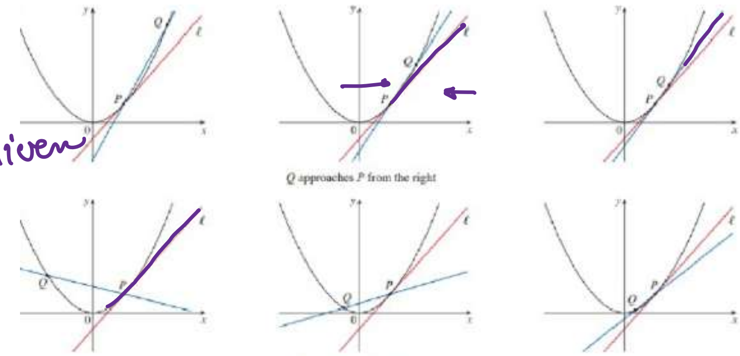
$y - 1 = 2x - 2$

$y = 2x - 2 + 1$

$\boxed{y = 2x - 1}$ ← معادلة الخط المستقيم المماس للمنحنى

$y = x^2$ at point $(1, 1)$

$\boxed{m = 2}$ وذلك بمعرفة القيمة التقريبية للميل

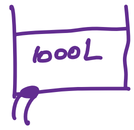


Q approaches P from the right

Q approaches P from the left

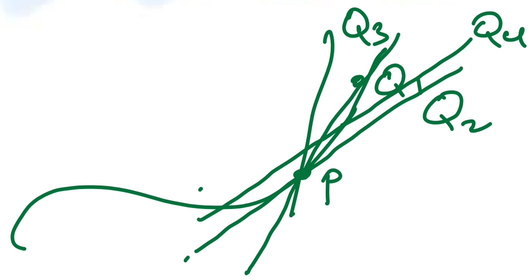
Exercise 2.1

1. A tank holds 1000 liters of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in liters) after t minutes.



	t_1	t_2	P			
x — t (min)	5	10	15	20	25	30
حجم الماء المتبقي في التانك — y — V (L)	694	444	250	111	28	0

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

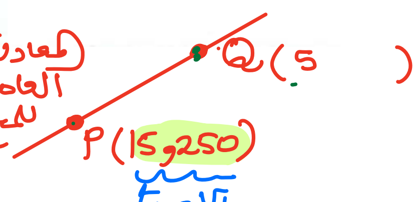


(a) If P is the point $(15, 250)$ on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25,$ and 30.

$t \to x \quad V \to y$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{V_2 - V_1}{t_2 - t_1}$$

المعادلة العامة للحل



$Q_1 \to t_2 = 5 \longrightarrow V_2 = 694 \longrightarrow m = \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{694 - 250}{5 - 15} = -44.4$ $\quad l_1$

$Q_2 \to t_2 = 10 \longrightarrow V_2 = 444 \longrightarrow m = \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{444 - 250}{10 - 15} = -38.8$ $\quad l_2$

$Q_3 \to t_2 = 20 \longrightarrow V_2 = 111 \longrightarrow m = \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{111 - 250}{20 - 15} = -27.8$ $\quad l_3$

$Q_4 \rightarrow t_2 = 25 \rightarrow v_2 = 28 \rightarrow m = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{28 - 250}{25 - 15} = -22.2$ l_4

$Q_5 \rightarrow t_2 = 30 \rightarrow v_2 = 0 \rightarrow m = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{0 - 250}{30 - 15} = -16.\dot{6}$ l_5

خمن

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

خمني ميل الخط المستقيم المماس عند النقطه P وذلك بايجاد متوسط
الميول لاثنى خطين قاطعين

The slope of the tangent line at P =

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$. $y = \frac{1}{1-x}$

(a) If Q is the point $(x, 1/(1 - x))$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 1.5	(ii) 1.9	(iii) 1.99	(iv) 1.999
(v) 2.5	(vi) 2.1	(vii) 2.01	(viii) 2.001

find the slope of the line $PQ \rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$

$$P(\underset{x_1}{2}, \underset{y_1}{-1}) \qquad Q(\underset{x_2}{x}, \underset{y_2}{\tfrac{1}{1-x}})$$

$$m = \frac{y_2 - y_1}{x_2 x_1} = \frac{\frac{1}{1-x} - (-1)}{x - 2} = \frac{\frac{1}{1-x} + 1 * \frac{1-x}{1-x}}{x-2}$$

$$= \frac{\frac{1}{1-x} + \frac{1(1-x)}{1-x}}{x-2} = \frac{\frac{1+1-x}{1-x}}{x-2}$$

$$= \frac{\frac{2-x}{1-x}}{x-2} = \frac{2-x}{1-x} \div x - 2$$

$$= \frac{2-x}{1-x} * \frac{1}{x-2} = \frac{2-x}{(1-x)(x-2)}$$

$$= \frac{2-x}{x - 2 - x^2 + 2x} = \frac{2-x}{-x^2 + 3x - 2}$$

$$m_{tan} = \frac{2-x}{-x^2 + 3x - 2}$$

الصورة العامة للميل
عند النقطة

$$P \quad Q \rightarrow$$
$$(2, -1) \quad \left(x, \frac{1}{1-x}\right)$$

نقطة ثابتة $P(2,-1)$

x	1.5	1.9	1.99	1.999	2.001	2.01	2.1	2.5	تتغير
m_{tan}	2	1.111111	1.0101	1.00100	0.999001	0.990	0.90909	0.666667	

$$m_{tan} = \frac{2-x}{-x^2+3x-2}$$

$$x=1.5 \to m = \frac{2-1.5}{-(1.5)^2+3(1.5)-2} = 2$$

$$x = 1.9 \to m = \frac{2-1.9}{-(1.9)^2+3(1.9)-2} = 1.11111$$

$$x = 1.99 \to m = \frac{2-1.99}{-(1.99)^2+3(1.99)-2} = 1.0101$$

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.

$\boxed{x=2}$

نلاحظ من الفقرة (a) انه كلما اقتربت قيمة x من 2

كلما اقترب الحل من $\boxed{1}$

we guess that slope at point $P(2,-1)$ will be $\boxed{1}$

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(\underset{x_1}{2}, \underset{y_1}{-1})$. $m = 1$

Equation

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = 1(x - 2)$$

$$y + 1 = x - 2$$

$$y = x - 2 - 1$$

$$\boxed{y = x - 3}$$

4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.

(a) If Q is the point $(x, \cos \pi x)$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 0 ✓ (ii) 0.4 ✓ (iii) 0.49

(iv) 0.499 ✓ (v) 1 (vi) 0.6

(vii) 0.51 (viii) 0.501

$$\overset{x_1}{}\ \overset{y_1}{} \qquad P(0.5, 0) \qquad\qquad Q(\overset{x_2}{x}, \overset{y_2}{\cos \pi x})$$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\cos \pi x - 0}{x - 0.5} = \frac{\cos \pi x}{x - 0.5}$$

الصيغة
العامة للميل

x	0	0.4	0.49	0.49	0.499	0.501	0.51	0.6	1
m	undefined								

$$x = 0 \longrightarrow m = \frac{\cos \pi x}{x - 0} = \frac{\cos \pi (0)}{0 - 0} \quad \text{undefined}$$

$$x = 0.4 \qquad m = \frac{\cos \pi (0.4)}{0.4 - 0} = ✓$$

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.

guess.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.

x_1 y_1

m =

$y - y_1 = m(x - x_1)$