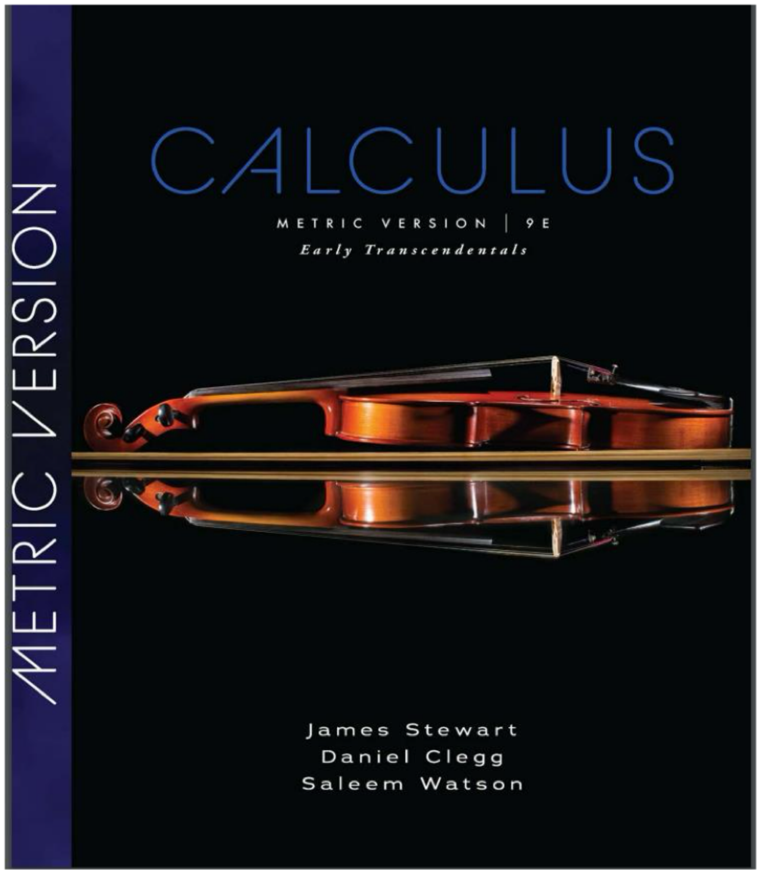
Calculus II, MATH 121



5 Integrals

5.1 The Area and Distance Problems

The Area Problem

We begin by attempting to solve the *area problem:* find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geqslant 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.

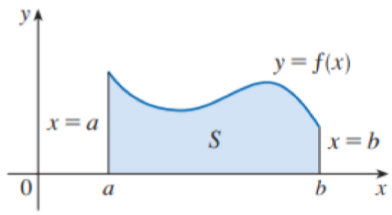
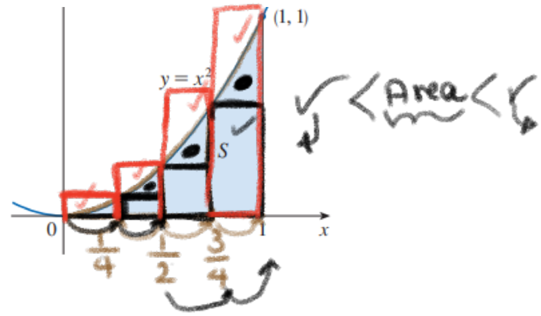
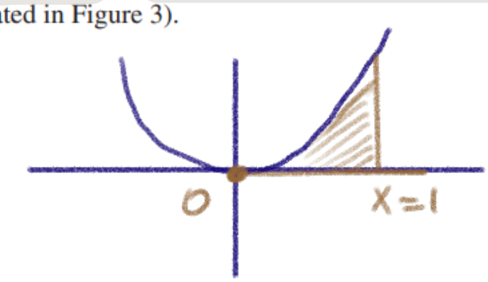
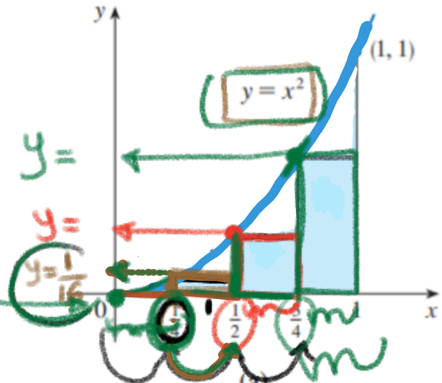


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

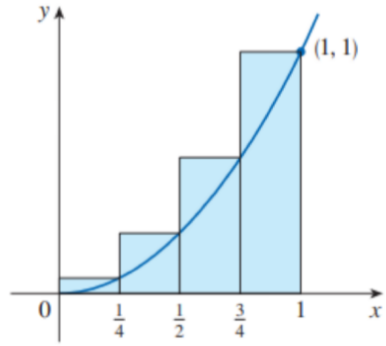
EXAMPLE 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).







(a)



(b)

x=0

$[0, \frac{1}{4}]$ → xx

$[\frac{1}{4}, \frac{1}{2}]$ → Leftend point of interval

$[\frac{1}{2}, \frac{3}{4}]$ → Left end point

$[\frac{3}{4}, 1]$ → Left end point

x^2 $\quad$ $y = x^2$

$$L_4 = \underbrace{\frac{1}{4}*f(0)}_{S_1} + \underbrace{\frac{1}{4}*f(\tfrac{1}{4})}_{S_2} + \underbrace{\frac{1}{4}*f(\tfrac{1}{2})}_{S_3} + \underbrace{\frac{1}{4}*f(\tfrac{3}{4})}_{S_5}$$

$$= \frac{1}{4}*0 + \frac{1}{4}*\frac{1}{16} + \frac{1}{4}*\frac{1}{4} + \frac{1}{4}*\frac{9}{16}$$

$$= \frac{1}{4}\left[0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right] = 0.21875$$

$$L_4 = \frac{1}{4}\left[\sum f(x)\right]$$

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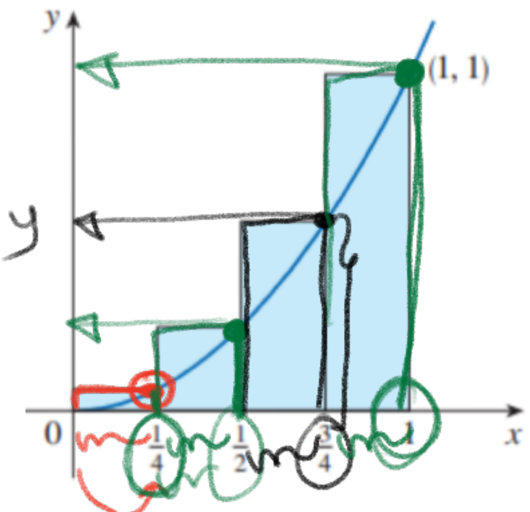
$\left[0, \frac{1}{4}\right]$ right end point

$\left[\frac{1}{4}, \frac{1}{2}\right]$ right end point

$\left[\frac{1}{2}, \frac{3}{4}\right]$ right

$\left[\frac{3}{4}, 1\right]$ right



(b)

$$\boxed{y = x^2}$$

$$R_4 = \frac{1}{4} * f\left(\frac{1}{4}\right) + \frac{1}{4} * f\left(\frac{1}{2}\right) + \frac{1}{4} * f\left(\frac{3}{4}\right) + \frac{1}{4} *$$

$$= \cancel{\frac{1}{4}} * \frac{1}{16} + \cancel{\frac{1}{4}} * \frac{1}{4} + \cancel{\frac{1}{4}} * \frac{9}{16} + \cancel{\frac{1}{4}} * 1$$

$$= \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right] = 0.46875$$

$$R_4 = \frac{1}{4} * \sum f(x$$

$$0.21875 < \text{Area} < 0.46875$$

Lower Sum

upper Sum

Lower Sum < Area < Upper Sum

L

Left [,] Right

عدد الاجزاء $* \sum f(x)$

R

Left [,] Right

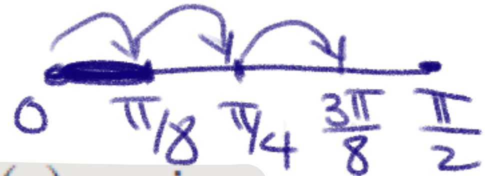
عدد الاجزاء $* \sum f(x)$

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

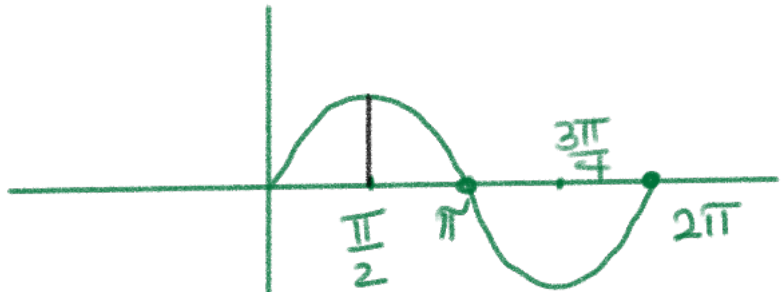
$A \approx 0.3333335.$



4. (a) Estimate the area under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi/2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

Right end points



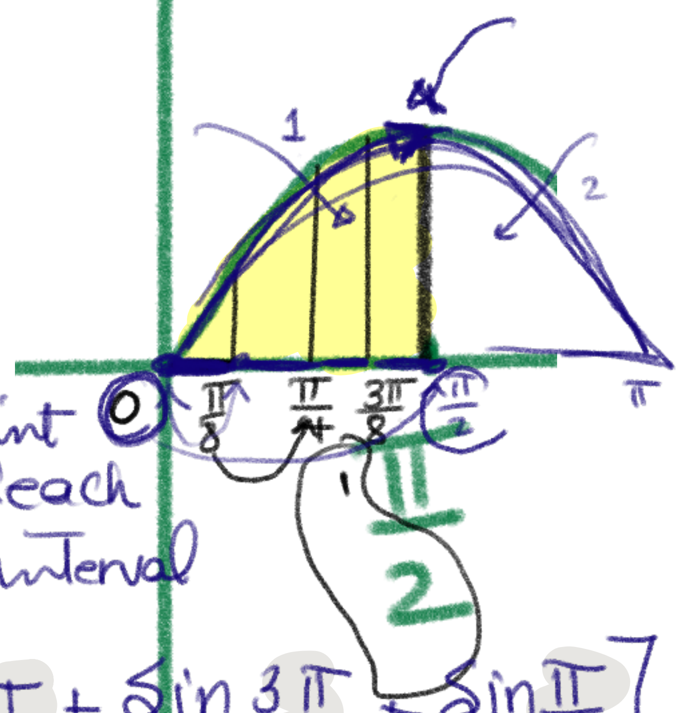
$\left[0, \frac{\pi}{8}\right]$

$\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$

$\left[\frac{\pi}{4}, \frac{3\pi}{8}\right]$

$\left[\frac{3\pi}{8}, \frac{\pi}{2}\right]$

$f(x) = y = \sin x$



right end point of each interval

$R_4 = \frac{\pi}{8} * \left[\Sigma f(x)\right]$

$= \frac{\pi}{8} * \left[\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2}\right]$

$= 1.1835 \longrightarrow > 1 \rightsquigarrow$ overestimating

Leftendpoint $f(x) = y = \sin x$

$[0, \frac{\pi}{8}]$

$[\frac{\pi}{8}, \frac{\pi}{4}]$

$[\frac{\pi}{4}, \frac{3\pi}{8}]$

$[\frac{3\pi}{8}, \frac{\pi}{2}]$

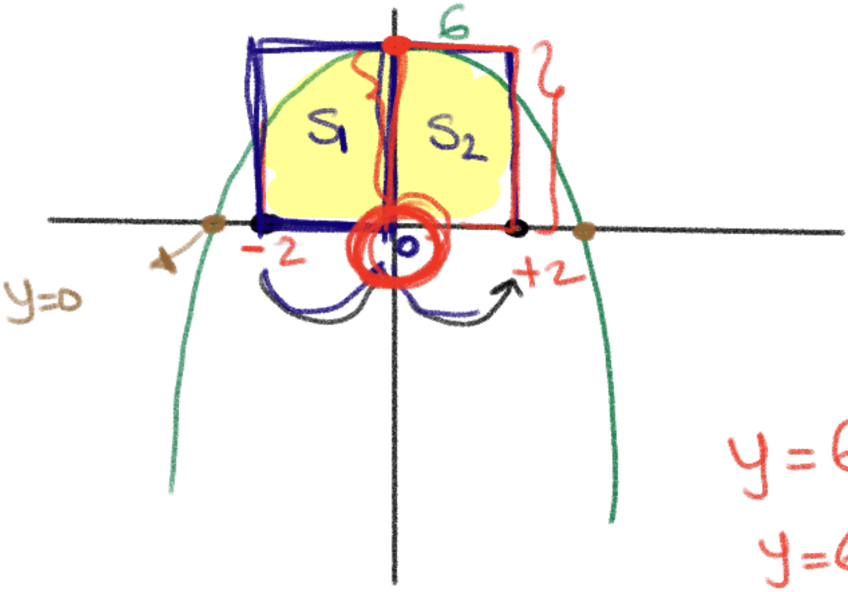
$$L_4 = \frac{\pi}{8} * \sum f(x)$$

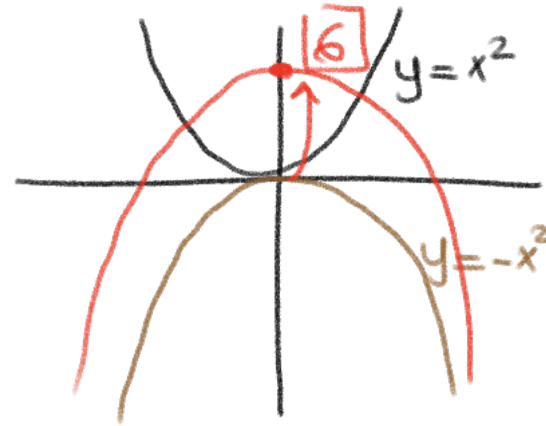
$$= \frac{\pi}{8}\left[\sin 0 + \sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8}\right]$$

$$= 0.7908 \rightarrow \underline{< 1} \text{ underestimate}$$

$$0.7908 < \text{Area} < 1.1835$$

7. Evaluate the upper and lower sums for $f(x) = 6 - x^2$, $-2 \leqslant x \leqslant 2$, with $n = 2, 4,$ and 8. Illustrate with diagrams like Figure 14.





$y = 6 - x^2$

$y = 6 - 0^2$

$0 = 6 - x^2$

$x^2 = 6$

$x = \pm\sqrt{6} = \pm 2\ldots$

$n = 2$

upper sum $= \Delta x \sum f(x)$

upper point

$= 2 * \sum f(x)$

$2 * f(0) + 2 * f(0)$

$= 2\left[f(0) + f(0)\right]$

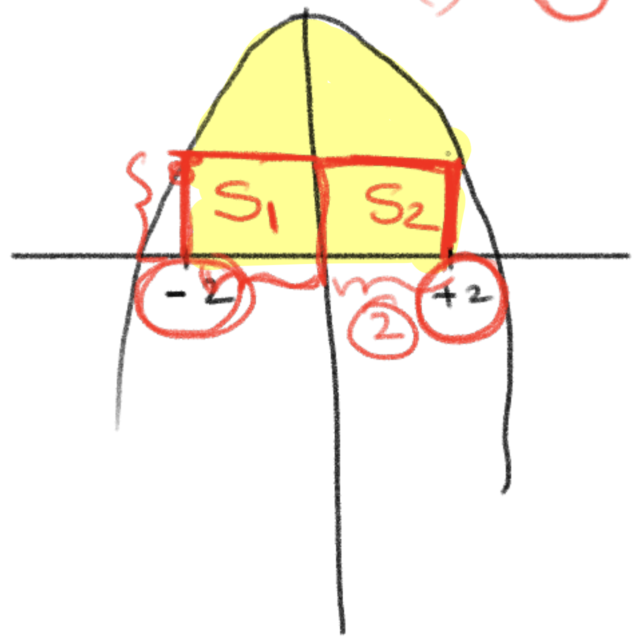
$= 2\left[6 + 6\right] = 24$

upper sum $= 24$

Lower sum $= \Delta x \sum f(x)$

$\hookrightarrow$ lower

$y = 6 - x^2$
$= 6 - (2)^2 = 2$



$2[f(-2) + f(2)]$

$L_2 = 2 * f(-2) + 2 * f(2)$

$= 2 * 2 + 2 * 2$

$= 4 + 4$

$= 8$

$\boxed{n=4}$

$f(x) = y = 6 - x^2$
$= 6 - (+1)^2$

uppersum $= \Delta x * \sum f(x)$

upper points

$1 * \sum f(x)$

$= 1 * [f(-1) + f(0) + f(0) + f(1)]$

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$= 1 [5 + 6 + 6 + 5]$

$= 22$

$y = 6 - x^2$

$(-1)^2$

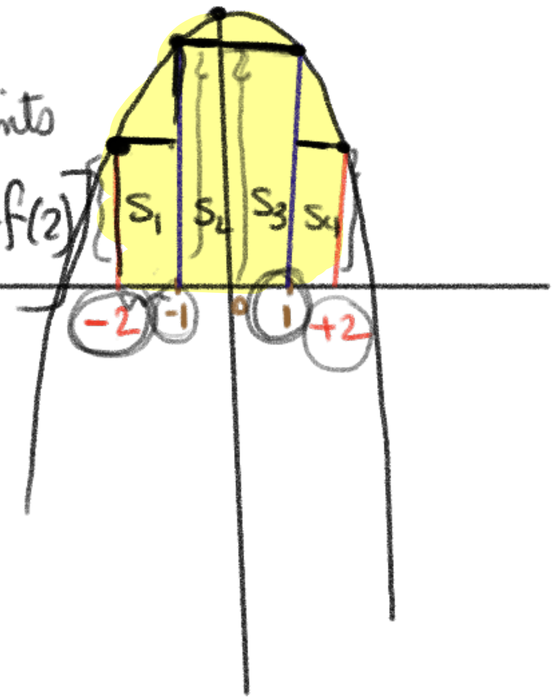
Lower Sum

$= \Delta x * \sum f(x)_{\to \text{Lower points}}$

$= 1\,[f(-2) + f(-1) + f(1) + f(2)]$

$= 1\,[2 + 5 + 5 + 2]$

$= 14$



\# $\boxed{n = 8}$ upper Sum = ---- → 205

Lower Sum ------ → 16.5

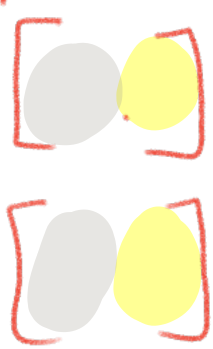
Estimate Area

upper sum

Lower sum

Right end point

Left end point



$$R = \text{Area} = \Delta x\left[\sum f(x_R)\right]$$

$$L = \text{Area} = \Delta x\left[\sum f(x_L)\right]$$