2.8 One-to-One Functions and Their Inverses

Objectives

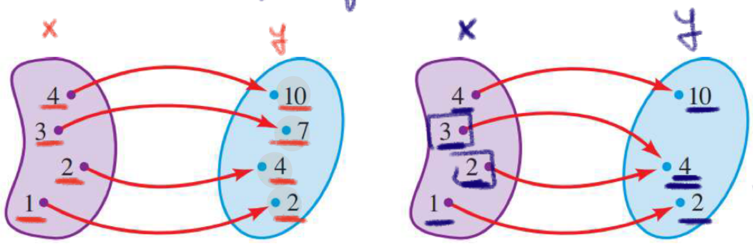
- One-to-One Functions
- The Inverse of a Function
- Finding the Inverse of a Function
- Graphing the Inverse of a Function
- Applications of Inverse Functions

DEFINITION OF A ONE-TO-ONE FUNCTION

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

$y_1 \neq y_2 \longrightarrow$



$x_1 = 3 \to f(x_1) - 4$

$x_2 = 2 \to f(x_2) = 4$

$x_1 \neq x_2 \leftarrow f(x_1) = f(x_2)$

$f(x_2) \neq f(x_1)$

$x_2 \neq x_1$

$x \to y$

one to one function

$x_1, x_2 \to y$

Not one to one function

HORIZONTAL LINE TEST

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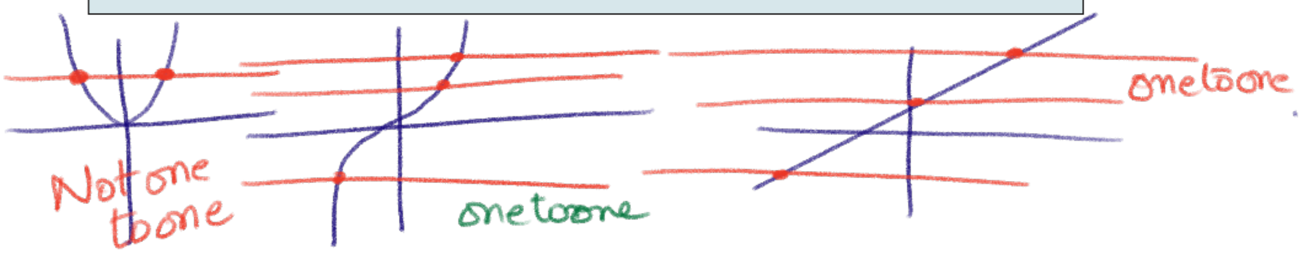
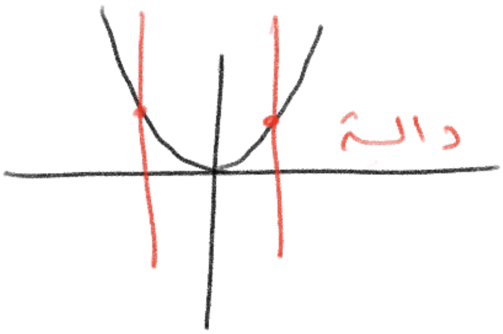
A function is one-to-one if and only if no horizontal line intersects its graph more than once.



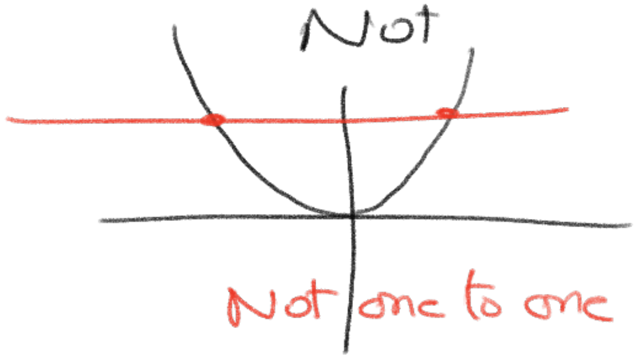
Vertical line test

Test whether the relation is a function or Not



Horizontal line test

Test whether the function is one to one or Not



EXAMPLE 1 ■ Deciding Whether a Function Is One-to-One

Is the function $f(x) = \boxed{x^3}$ one-to-one?

If $x_1 \neq x_2 \longrightarrow f(x_1) = (x_1)^3 = x_1^3$

$f(x_2) = (x_2)^3 = x_2^3$

$f(x_1) \neq f(x_2) \longrightarrow$ one to one function

If $x_1 = -1 \neq x_2 = 1$

$f(x_1) = f(-1) = (-1)^3 = -1$

$f(x_2) = f(1) = (1)^3 = 1$

$\neq$ one to one function

If $f(x_1) = f(x_2)$

$x_1^3 = x_2^3 \longrightarrow x_1 \neq x_2$ one to one function

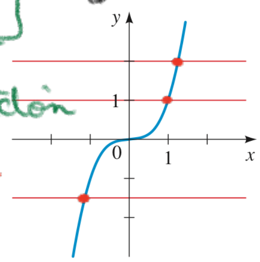


FIGURE 3 $f(x) = x^3$ is one-to-one.

EXAMPLE 2 ■ Deciding Whether a Function Is One-to-One

Is the function $g(x) = x^2$ one-to-one?

$x_1 = 1$ $\qquad$ $x_2 = -1$

$g(x_1) = (1)^2 = 1$ $\qquad$ $g(x_2) = (-1)^2 = 1$

$x_1 \neq x_2$

$g(x_1) = g(x_2)$ Not one to one function

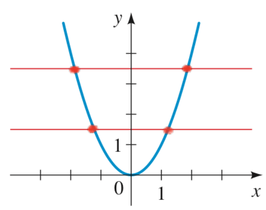


FIGURE 4 $g(x) = x^2$ is not one-to-one.

EXAMPLE 3 ■ Showing That a Function Is One-to-One

Show that the function $f(x) = 3x + 4$ is one-to-one.

ط ط $y = 3x + 4$

$x = 0 \longrightarrow y = 3(0) + 4 = 4$

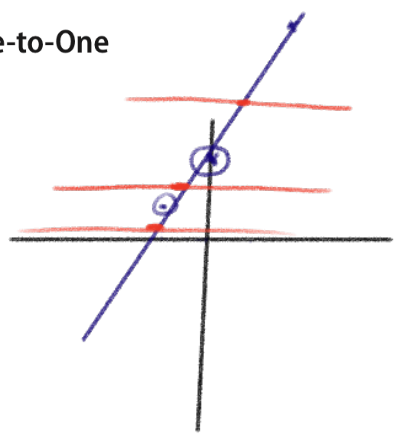
$x = 1 \longrightarrow y = 3(1) + 4 = 7$

$x = -1 \longrightarrow y = 3(-1) + 4 = 1$

$\neq$

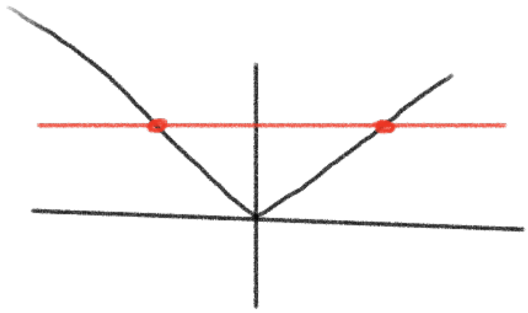
$x_1 \neq x_2 \qquad f(x_1) \neq f(x_2)$

one to one function



13–24 ■ One-to-One Function? Determine whether the function is one-to-one.

16. $g(x) = |x|$



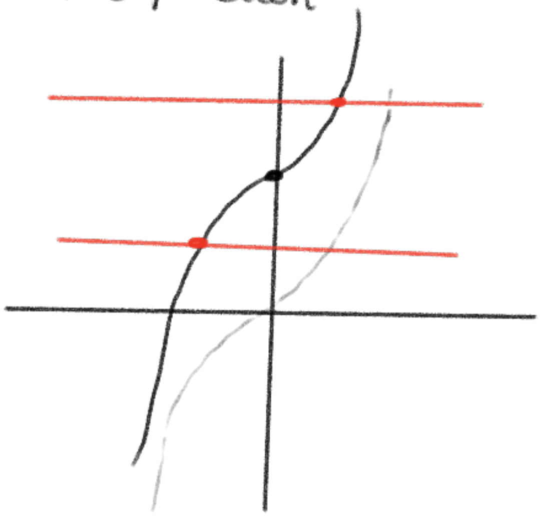
$x_1 = 1 \neq x_2 = -1$

$f(x_1) = f(1) = |1| = 1$

$f(x_2) = |-1| = 1$

$x_1 \neq x_2 \rightarrow f(x_1) = f(x_2)$ Not one to one function

18. $h(x) = x^3 + 8$



$x_1 = 1$ $x_2 = -1$

$f(x_1) = f(1) = (1)^3 + 8 = 9$

$f(x_2) = (-1)^3 + 8 = 7$

$x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$

one to one function

24. $f(x) = \dfrac{1}{x}$

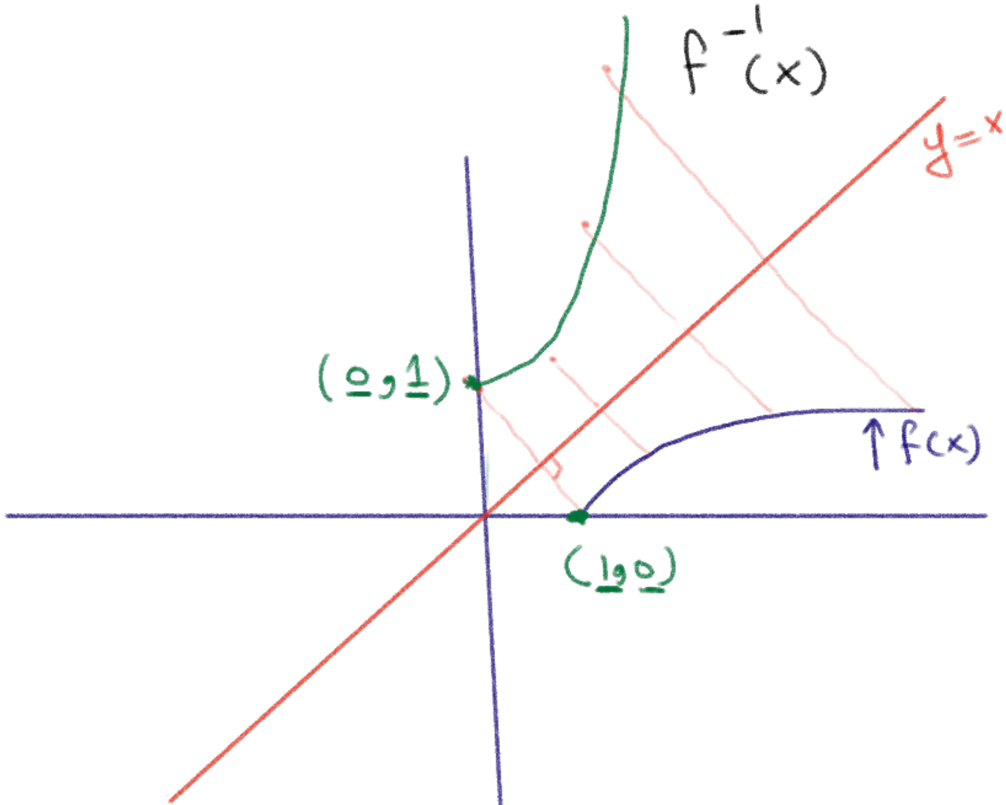
$x_1 = 1 \neq x_2 = -1$

$f(x_1) = \frac{1}{1} = 1$

$f(x_2) = \frac{1}{-1} = -1$

$\neq$

one to one function



$(2,4)\quad(3,9)\quad(4,16) \longrightarrow f(x)$

$(4,2)\quad(9,3)\quad(16,4) \quad f^{-1}(x)$

$f(2)=4$ $\qquad$ $f(3)=9$ $\qquad$ $f(4)=16$

$f^{-1}(4)=2$ $\qquad$ $f^{-1}(9)=3$ $\qquad$ $f^{-1}(16)=4$

EXAMPLE 4 ■ Finding f^{-1} for Specific Values

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(-10)$.

$f(1) = 5 \longrightarrow f^{-1}(5) = 1$

$f(3) = 7 \longrightarrow f^{-1}(7) = 3$

$f(8) = -10 \longrightarrow f^{-1}(-10) = 8$

25–28 ■ Finding Values of an Inverse Function Assume that f is a one-to-one function.

26. **(a)** If $f(5) = 18$, find $f^{-1}(18) = 5$

(b) If $f^{-1}(4) = 2$, find $f(2) = 4$

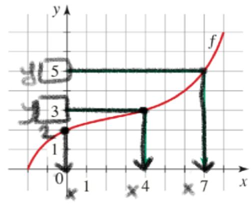
EXAMPLE 5 ■ Finding Values of an Inverse Function

We can find specific values of an inverse function from a table or graph of the function itself.

x	$h(x)$
2	5
3	8
4	12
5	1
6	3
7	15

$f^{-1}(5)=2$
$f^{-1}(8)=3$
$f^{-1}(12)=4$
$f^{-1}(1)=5$
$f^{-1}(3)=6$
$f^{-1}(15)=7$

Finding values of h^{-1} from a table of h



$f^{-1}(5)=7$
$f^{-1}(3)=4$
$f^{-1}(2)=0$

FIGURE 8 Finding values of f^{-1} from a graph of f

31–36 ■ Finding Values of an Inverse Using a Table

A table of values for a one-to-one function is given. Find the indicated values.

34. $f(f^{-1}(6))$

x	1	2	3	4	5	6
$f(x)$	$\boxed{4}$	6	2	5	0	1

$f(1)=4$ $\quad \boxed{f(2)=6}$ $\quad f(3)=2$ $\quad f(4)=5$ $\quad f(5)=0$ $\quad f(6)=1$

$f^{-1}(4)=1$ $\quad \boxed{f^{-1}(6)=2}$ $\quad f^{-1}(2)=3$ $\quad f^{-1}(5)=4$ $\quad f^{-1}(0)=5$ $\quad f^{-1}(1)=6$

$f(\underbrace{f^{-1}(6)}) = f(2) = 6$

$$\boxed{\begin{array}{l} f f^{-1}(x) = x \\ f^{-1} f(x) = x \end{array}}$$

$$f f^{-1}(2) = 2$$

$$f^{1} f^{-1}(-1) = -1$$

$$f^{-1} f(2) = 2$$

$$f^{-1} f^{1}(5) = 5$$