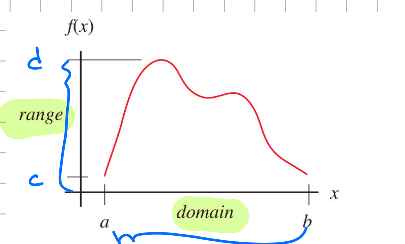
4.5 Summary of Curve Sketching

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.)

But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

[c , d]



[a, b]

Domain

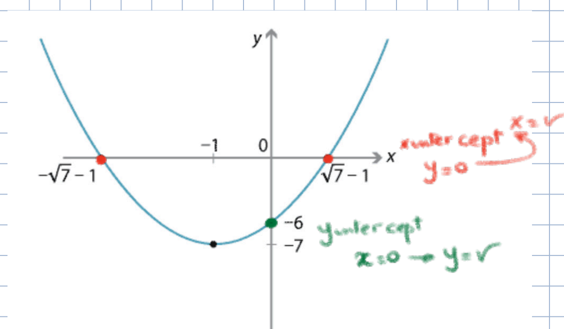
$y = ax + b \rightarrow R$

$y = ax^2 + bx + c \rightarrow R$ ∪ ∩

$y = \frac{\checkmark}{\checkmark} \rightarrow R - \{$[illegible]$\}$

$y = \sqrt{} \rightarrow \bigcirc \geq 0$

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)



C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an *even function* and the curve is symmetric about the y-axis.

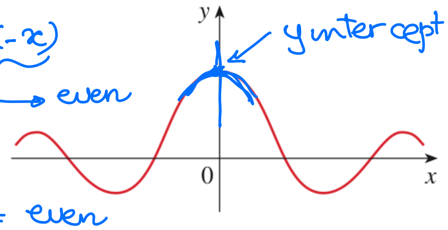
$x \to -x \to f(-x)$

$f(-x) = f(x) \to$ even

$y = x^2 + 4$

$f(-x) = (-x)^2 + 4$

$f(-x) = x^2 + 4$ = even



y intercept $\to x = 0$

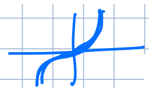
(a) Even function: reflectional symmetry

Cos $\to$ even زوجية

متماثلة حول محور y

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an *odd function* and the curve is symmetric about the origin.

$f(x) = x^3$

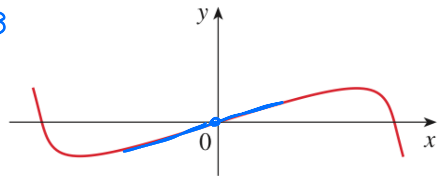


$f(-x) = (-x)^3 = -x^3$

$-f(x) = -x^3$

$f(-x) = -f(x)$

$f(x)$ odd



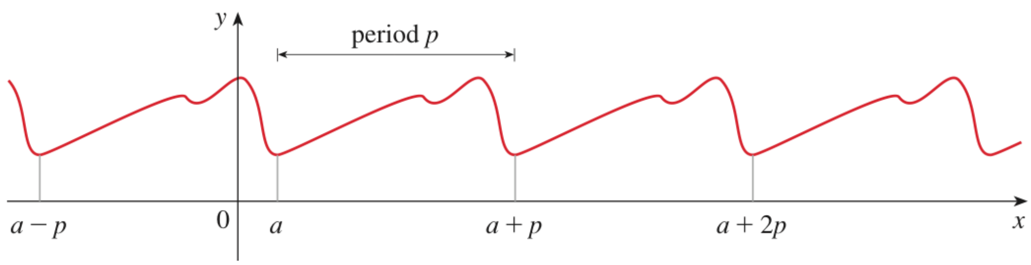
(b) Odd function: rotational symmetry

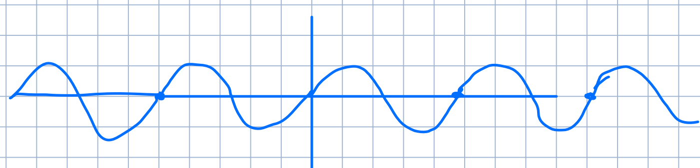
Sin

الدوال الفردية متماثلة

حول origin

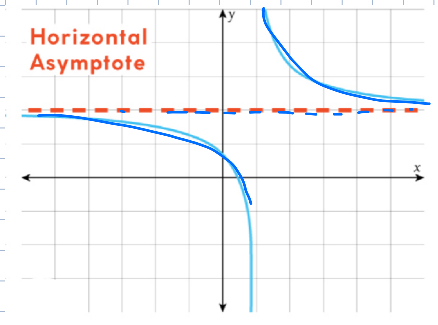
(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is a **periodic function** and the smallest such number p is called the **period**.





D. Asymptotes

(i) *Horizontal Asymptotes*. Recall from Section 2.6 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.



Horizontal Asymptote

قيمة الناتج للدالة

$y = L$

$\lim_{x\to\pm\infty} f(x) = L$

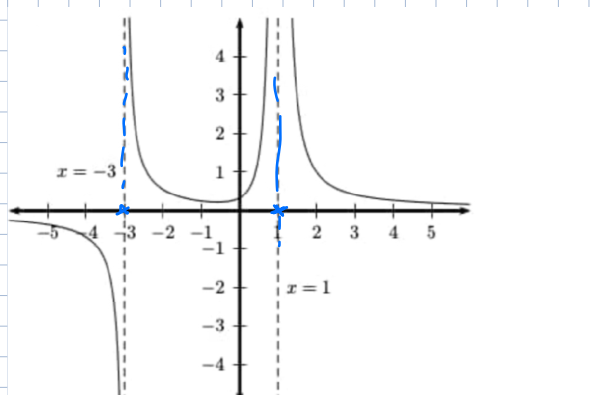
rational function

(ii) *Vertical Asymptotes*. Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\boxed{1} \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.



Rational function

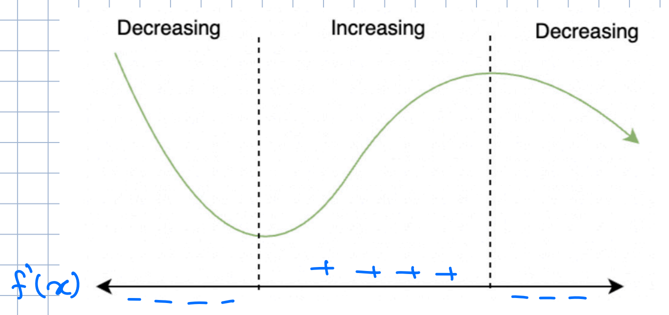
$\frac{a}{b}$

Denomenator = zero.

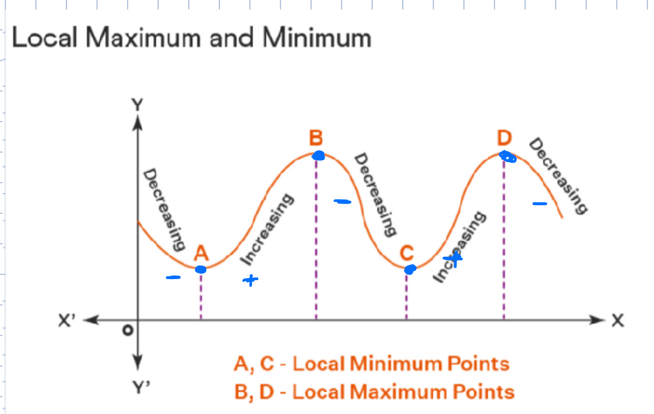
$x =$ ✓ $\qquad x =$ ✓

(iii) *Slant Asymptotes*. These are discussed at the end of this section. xx

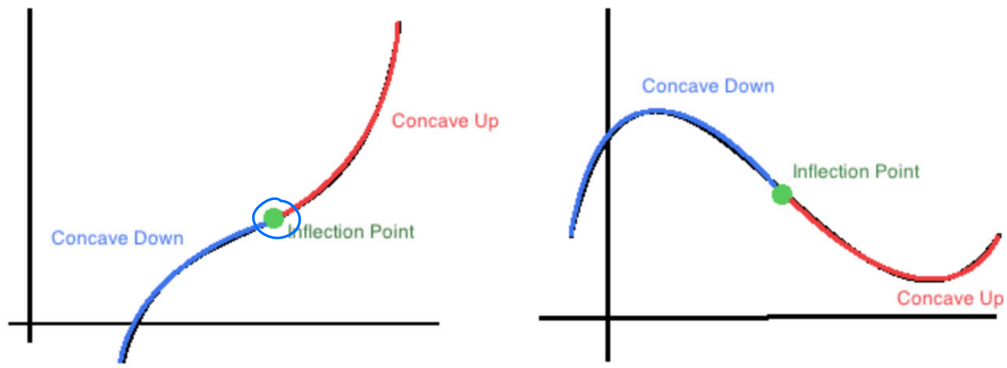
E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).



F. Local Maximum or Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.



G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.



H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

Use the guidelines to sketch the curve $y = x^4 + 4x^3$ ← polynomial

A. Domain : all real number

B. Intercepts

x intercept → $y=0$ → $0 = x^4 + 4x^3$

$x^3[x+4]=0$

$x^3=0$ or $x+4=0$

$x=0$ $\quad$ $x=-4$

$(0,0)$ $(-4,0)$

y intercept → $x=0$ → $y=0^4+4(0)^3=0$

$(0,0)$

Symmetry → $f(x) = y = x^4 + 4x^3$

$f(-x) = (-x)^4 + 4(-x)^3$

$f(-x) = x^4 - 4x^3$

$f(-x) \neq f(x)$ → Not even

$-f(x) = -[x^4 + 4x^3] = -x^4 - 4x^3$

$-f(x) \neq f(-x)$ → Not odd

No Symmetry in this function

$$y = x^4 + 4x^3$$

D. Asymptotes

$\longrightarrow$ There is no asymptote in polynomials

Vertical

مقام = صفر

لا يوجد مقام

Horizontal

$\lim\limits_{x \to \infty} f(x) = L$ مش رقم

$\lim\limits_{x \to \infty} f(x) = \infty^4 + 4(\infty)^3 = \infty$

E. Intervals of Increase or Decrease

$$y = x^4 + 4x^3$$

$f'(x) = 4x^3 + 12x^2$

$f'(x) = 0 \longrightarrow$ Critical values

$4x^3 + 12x^2 = 0 \longrightarrow 4x^2[x + 3] = 0$

$\frac{4x^2}{4} = \frac{0}{4}$ or $x + 3 = 0$

$x = 0$ $\qquad$ $x = -3$

$-4 \qquad -3 \qquad -1 \qquad 0 \qquad +1$

$4x^2$ $\quad$ $4(-4)^2 = +$ $\quad$ $4(-1)^2 = +$ $\quad$ $4(1)^2 = +$

$(x+3)$ $\quad$ $(-4+3)$ $-$ $\quad$ $-1+3$ $+$ $\quad$ $1+3 = +$

$(-\infty, -3)$ decreasing function

$(-3, \infty)$ increasing function

F. Local Maximum or Minimum Values

Change of sign of $f'(x$

$- \quad + \quad \longrightarrow$ Local minimum at $x=-3$

$y = x^4 + 4x^3 = (-3)^4 + 4(-3)^3 = 81 - 108 = -27$

Local minimum at point $(-3,-27)$

G. Concavity and Points of Inflection

$y = x^4 + 4x^3$

$f'(x) = 4x^3 + 12x^2$

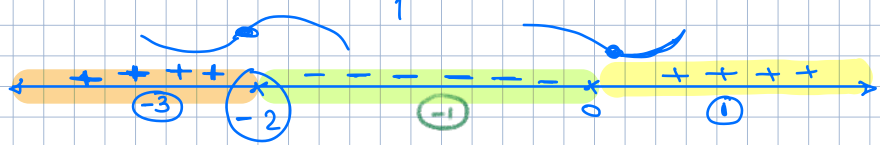
$f''(x) = 12x^2 + 24x$

$f''(x) = 0 \longrightarrow 12x^2 + 24x = 0$

$12x[x+2] = 0$

$12x = 0$ or $x + 2 = 0$

$x = 0$ $\qquad$ $x = -2$



$12x$	$12(-3)$ $-$	$12(-1) = -$	$12(1) = +$
$x+2$	$-3+2 = -$	$-1+2$ $+$	$1+2 = +$

$(-\infty,-2) \longrightarrow f''(x) > 0 \longrightarrow$ Concave up.

$(-2,0) \longrightarrow f''(x) < 0 \longrightarrow$ Concave down

$(0,\infty) \longrightarrow f''(x) > 0 \longrightarrow$ Concave up

Inflection points:

$$y = x^4 + 4x^3$$

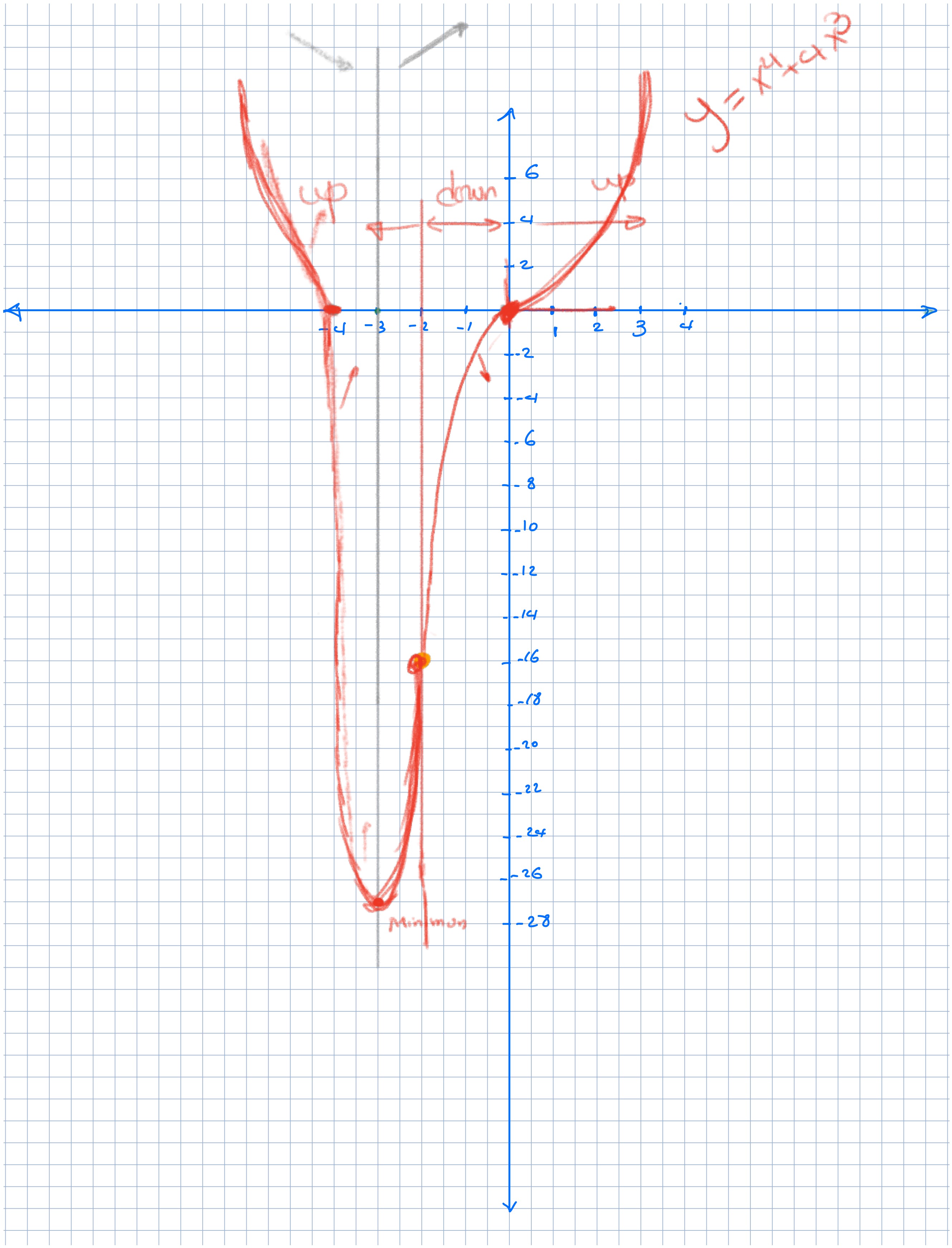
$x = -2 \longrightarrow y = (-2)^4 + 4(-2)^3 = 16 - 32 = -16$

$(-2, -16)$

$x = 0 \longrightarrow y = (0)^4 + 4(0)^3 = 0$

$(0, 0)$

Domain	all real number $(-\infty, \infty)$
Intercepts	$(0,0)$ $(-4,0)$ $(0,0)$ احداثيات نقاط التقاطع
Vertical asymptote	No
Horizontal asymptote	No
Interval increase	$(-3, \infty)$
Interval decrease	$(-\infty, -3)$
Local Maximum	No
Local Minimum	$(-3, -27)$
Concave up	$(-\infty, -2)$ $(0, \infty)$
Concave down	$(-2, 0)$
Inflection point	$(-2, -16)$, $(0, 0)$



y = x^4+4x^3
up
down
up
Minimum
6
4
2
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
-26
-28
-4
-3
-2
-1
1
2
3
4

1–54 Use the guidelines of this section to sketch the curve.

H3

1. $y = x^3 + 3x^2$

A. Domain

B. Intercepts

C. Symmetry

D. Asymptotes

E. Intervals of Increase or Decrease

F. Local Maximum or Minimum Values

G. Concavity and Points of Inflection

H. Inflection point

Domain	
Intercepts	
vertical asymptote	
horizontal asymptote	
Interval increase	
Interval decrease.	
Local Maximum	
Local Minimum	
Concave up	
Concave down	
Inflection point	