

3.6

Derivatives of Logarithmic and Inverse Trigonometric Functions

Derivatives of Logarithmic Functions

$$f(x) = \ln x \text{ or } f(x) = \log_b x$$

$$f'(x) = \frac{1}{x} \quad f'(x) = \frac{1}{x \ln b}$$

1

$$\frac{d}{dx} (\log_b x) = \frac{1}{x \ln b}$$

2

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

3

$$\frac{d}{dx} (\ln u) = \frac{1}{u} \frac{du}{dx}$$

or

$$\frac{d}{dx} [\ln g(x)] = \frac{g'(x)}{g(x)}$$

Ex: $x^2 + 3x + 5$

$\ln(x^2 + 3x + 5)$

$\frac{1}{u} \cdot \frac{du}{dx}$

$\frac{1}{x^2 + 3x + 5} \cdot (2x + 3)$

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

$$y' = \frac{1}{x^3+1} * (3x^2 + 0) = \frac{3x^2}{x^3+1}$$

2. $g(t) = \ln(3 + t^2)$

$$g'(t) = \frac{1}{3+t^2} * (0 + 2t) = \frac{2t}{3+t^2}$$

4. $f(x) = x \ln x - x$

Diagram showing the product rule for $x \ln x$:
① x (labeled 1) and ② $\ln x$ (labeled 2) are circled.
An arrow points from ① to ② with the label "product rule".
Below the arrow, the expression $x \ln x$ is written with a bracket underneath it.

$$\textcircled{1} \textcircled{2}' + \textcircled{2} \textcircled{1}'$$

$$f'(x) = \underbrace{x * \frac{1}{x}}_{\textcircled{1} \textcircled{2}'} + \ln x * 1_{\textcircled{2} \textcircled{1}'} - 1$$

$$f'(x) = \cancel{1} + \ln x - \cancel{1} = \ln x$$

EXAMPLE 2 Find $\frac{d}{dx} \ln(\sin x)$. $= \frac{1}{\sin x} * \cos x = \frac{\cos x}{\sin x} = \cot x$

5. $f(x) = \sin(\ln x) = \cos(\ln x) * \frac{1}{x} = \frac{\cos(\ln x)}{x}$

Annotations: Arrows point from 'sin' to 'cos' and from 'ln x' to 'x'. Below 'ln x' is the text 'ln x'.

16. $P(v) = \frac{\ln v}{1-v} \rightarrow$ quotient rule

$$P'(v) = \frac{\frac{1}{v} * (1-v) - \ln v * (-1)}{(1-v)^2}$$

*Annotations: An arrow points from '1/v' to '1/v * (1-v)'. Another arrow points from 'ln v * (-1)' to 'ln v * [0-1]'. A checkmark is to the right.*

$$= \frac{\frac{1-v}{v} + \ln v}{(1-v)^2}$$

Annotation: An arrow points to the right with the text 'answer'.

$$(\textcircled{1}\textcircled{2}' + \textcircled{2}\textcircled{1}')$$

22. $g(x) = e^{x^2 \ln x}$

$$g'(x) = e^{x^2 \ln x} * \left[x^2 * \frac{1}{x} + \ln x * 2x \right]$$

$$= e^{x^2 \ln x} [x + 2x \ln x]$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x} = (\ln x)^{1/2}$ → power rule

① انقوس → $\frac{1}{2} - 1$

$$f'(x) = \frac{1}{2} (\ln x)^{-1/2} * \frac{1}{x}$$

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$$= \frac{(\ln x)^{-1/2}}{2x} = \frac{1}{2x(\ln x)^{1/2}} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

$$f'(x) = \frac{1}{(2 + \sin x) \ln 10} * [\cancel{0} + \cos x]$$

$$= \frac{\cos x}{(2 + \sin x) \ln 10}$$

$$\log_b u \rightarrow \frac{1}{u \ln b} * u'$$

EXAMPLE 5 Find $\frac{d}{dx} \ln \left(\frac{x+1}{\sqrt{x-2}} \right)$ $\rightarrow \frac{1}{\ln} * \frac{\text{derivative of } \ln}{\ln}$

$$= \ln(x+1) - \ln \sqrt{x-2}$$

$$= \ln(x+1) - \ln(x-2)^{1/2}$$

$$f(x) = \ln(x+1) - \frac{1}{2} \ln(x-2)$$

$\downarrow$ derivative

$$f'(x) = \frac{1}{x+1} * [\cancel{1+0}] - \frac{1}{2} * \frac{1}{(x-2)} * [\cancel{1-0}]$$

$$= \frac{1}{x+1} - \frac{1}{2(x-2)}$$

$\ln$ is additive
 $\downarrow$

$$\ln \frac{P}{M} = \ln P - \ln M$$

$$\ln P \cdot M = \ln P + \ln M$$

$$\ln M^P = P \ln M$$

EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$. $\downarrow$ 0 $\frac{d}{dx}$

$$f(x) = \begin{cases} \ln(x) & x > 0 \rightarrow f'(x) = \left(\frac{1}{x}\right) \\ \ln(-x) & x < 0 \rightarrow f'(x) = \frac{1}{-x} \cdot -1 \\ & = \left(\frac{1}{x}\right) \end{cases}$$

$$f'(x) = \frac{1}{x}$$

$$x \neq 0$$

29–32 Find y' and y'' .
 $\frac{d}{dx}$ $\frac{d^2}{dx^2}$

31. $y = \ln|\sec x|$

$$y = \begin{cases} \ln(\sec x) \rightarrow y' = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x \\ y'' = \sec x \tan x \end{cases}$$

$$\ln(-\sec x) \rightarrow y' = \frac{1}{-\sec x} \cdot -\sec x \tan x = \tan x$$

$$y'' = \sec x \tan x$$

$$y' = \tan x \text{ \& } y'' = \sec x \tan x.$$

39-40 Find an equation of the tangent line to the curve at the given point.

40. $y = x^2 \ln x$, $(1, 0)$

$$y - y_1 = m(x - x_1)$$

حيث x_1, y_1 هي إحداثيات نقطة التماس

$$m_{\text{tan}} = y'$$

$$y' = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x$$

$$y' = x + 2x \ln x$$

$$m_{\text{tan}} = y'|_{x=1} = 1 + 2 \cdot 1 \cdot \ln 1 = 1$$

$$\boxed{m_{\text{tan}} = 1}$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 1(x - 1)$$

$$\boxed{y = x - 1}$$

