7.3 Double-Angle, Half-Angle, and Product-Sum Formulas

Objectives

- Double-Angle Formulas
- Half-Angle Formulas
- Evaluating Expressions Involving Inverse Trigonometric Functions
- Product-Sum Formulas

Double-Angle, Half-Angle, and Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **Double-Angle Formulas** allow us to find the values of the trigonometric functions at $2x$ from their values at x.

The **Half-Angle Formulas** relate the values of the trigonometric functions at $\frac{1}{2}x$ to their values at x. The **Product-Sum Formulas** relate products of sines and cosines to sums of sines and cosines.

DOUBLE-ANGLE FORMULAS

$[2(\quad)(\quad)]$

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

→ $\sin 2x = \checkmark$

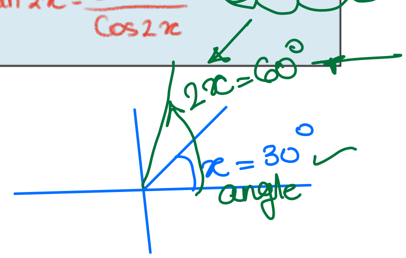
→ $\cos 2x = \checkmark$

Formula for tangent: $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$ or $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$

double angle.

$x \to 2x$

$30 \to 2 * 30 = 60°$



$x = 30$

$2x = 60$

$$\sin 2x = 2\sin x \cos x$$

$$\sin(60) = 2\sin 30 \cos 30$$

$$\frac{\sqrt{3}}{2} = \cancel{2} * \frac{1}{\cancel{2}} * \frac{\sqrt{3}}{2}$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

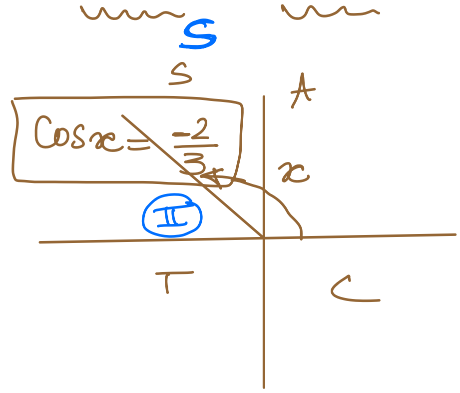
sin | cos | tan

$\boxed{2x}$

$\sin x \quad \cos x \quad \tan x$

EXAMPLE 1 ■ Using the Double-Angle Formulas

If $\cos x = -\frac{2}{3}$ and x is in Quadrant II, find $\cos 2x$ and $\sin 2x$.



$$\cos 2x = 2\cos^2 x - 1$$
$$= 2\left(-\frac{2}{3}\right)^2 - 1$$
$$= 2\left(\frac{4}{9}\right) - 1$$
$$= \frac{8}{9} - \frac{9}{9} = \frac{-1}{9}$$

$$\boxed{\cos 2x = -1/9}$$

$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \quad \text{or} \quad \tan 2x = \frac{\sin 2x}{\cos 2x}$$

$\rightarrow \sin 2x = \checkmark$
$\rightarrow \cos 2x = \checkmark$

$\sin 2x = 2\sin x \cos x$
$$= 2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{-2}{3}\right)$$

$$\boxed{\sin 2x = \frac{-4\sqrt{5}}{9}}$$

Remember

$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x = 1 - \cos^2 x$$
$$\sin x = \pm\sqrt{1 - \cos^2 x}$$
$$\sin x = +\sqrt{1 - \cos^2 x}$$

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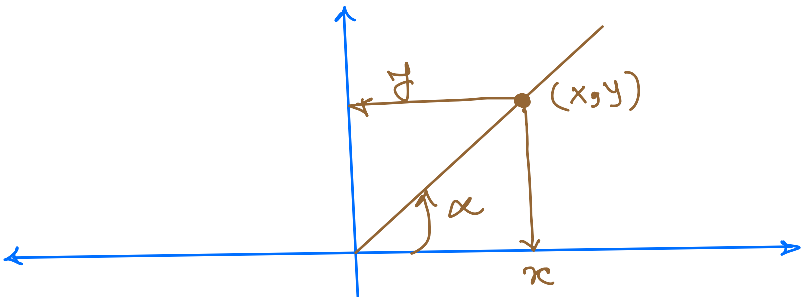
$$\sin x = \sqrt{1 - \left(-\frac{2}{3}\right)^2}$$
$$= \sqrt{1 - \frac{4}{9}}$$
$$= \sqrt{\frac{9}{9} - \frac{4}{9}} = \sqrt{\frac{5}{9}}$$
$$\sin x = \frac{\sqrt{5}}{3}$$



$\sin\alpha = \frac{y}{r}$

$\cos\alpha = \frac{x}{r}$

$\tan\alpha = \frac{y}{x}$

$\csc\alpha = \frac{r}{y}$

$\sec\alpha = \frac{r}{x}$

$\cot\alpha = \frac{x}{y}$

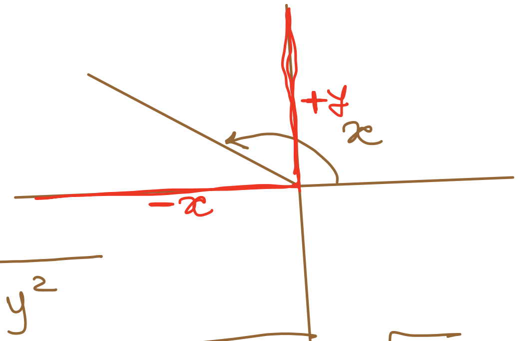
$r = \sqrt{x^2 + y^2}$

$x = \pm\sqrt{r^2 - y^2}$

$y = \pm\sqrt{r^2 - x^2}$

3–10 ■ Double Angle Formulas Find $\sin 2x$, $\cos 2x$, and $\tan 2x$ from the given information.

4. $\tan x = -\frac{4}{3}$, x in Quadrant II



$$\tan x = \frac{-4}{3} = \boxed{\frac{y}{x} = \frac{4}{-3}}$$

$$x = -3 \qquad y = 4 \qquad r = \sqrt{x^2+y^2}$$

$$= \sqrt{(-3)^2+(4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\boxed{x = -3 \qquad y = 4 \qquad r = 5}$$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$ → $\sin 2x = $ ✓

$= 1 - 2\sin^2 x$ → $\cos 2x = $ ✓

$= 2\cos^2 x - 1$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$ or $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$

[1] $\sin 2x = 2 \sin x \cos x$

$$= 2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$

$$= 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) = \boxed{\frac{-24}{25}}$$

[2] $\cos 2x = 1 - 2\sin^2 x$

$$= 1 - 2\left(\frac{y}{r}\right)^2 = 1 - 2\left(\frac{4}{5}\right)^2 = 1 - 2\left(\frac{16}{25}\right)$$

$$= 1 - \frac{32}{25} = \frac{25}{25} - \frac{32}{25} = \boxed{\frac{-7}{25}}$$

H.w $\cos 2x = 2\cos^2 x - 1$ $\qquad$ $\cos 2x = \cos^2 x - \sin^2 x$

$\boxed{3}$ $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$$= \frac{2\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)^2} = \frac{2\left(\frac{-4}{3}\right)}{1-\left(\frac{-4}{3}\right)^2}$$

$$= \frac{-8/3}{1-(16/9)} = \frac{-8/3}{9/9 - 16/9}$$

$$= \frac{-8/3}{-7/9} = \frac{-8}{3} \div \frac{-7}{9}$$

$$= \frac{-8}{\cancel{3}_1} \times \frac{-\cancel{9}^{3}}{7} = \boxed{\frac{24}{7}}$$

$$\boxed{\tan 2x = \frac{24}{7}}$$

Method 2// to find $\tan 2x = \dfrac{\sin 2x}{\cos 2x} = \dfrac{-24/25}{-7/25}$

$$= \frac{-24}{25} \div \frac{-7}{25} = \frac{-24}{\cancel{25}} \times \frac{-\cancel{25}}{7} = \boxed{\frac{24}{7}}$$

6. $\csc x = 4$, $\tan x < 0$

معلوم س sin ⊕   tan سالبة

$+\sin$
$\ominus \tan$

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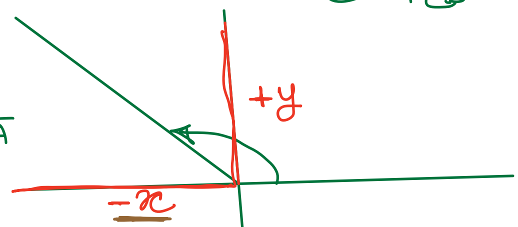
الربع الثاني



Given

$\csc x = \frac{4}{1} = \frac{r}{y}$ → $\sin x = \frac{1}{4}$

$r = 4 \qquad y = 1 \qquad x = \pm\sqrt{r^2 - y^2}$

$= -\sqrt{4^2 - 1^2} = -\sqrt{16 - 1} = -\sqrt{15}$

$\sin = \frac{y}{r}$

$\csc = \frac{r}{y}$

$$\boxed{x = -\sqrt{15} \qquad y = 1 \qquad r = 4}$$

$\sin 2x = 2\sin x \cos x$

$= 2\left[\frac{y}{r}\right]\left[\frac{x}{r}\right]$

$= \cancel{2}\left[\frac{1}{\cancel{4}_2}\right]\left[\frac{-\sqrt{15}}{4}\right] = \frac{-\sqrt{15}}{8}$

$$\boxed{\sin 2x = \frac{-\sqrt{15}}{8}}$$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$ → $\sin 2x = ✓$

$= 1 - 2\sin^2 x$ → $\cos 2x = ✓$

$= 2\cos^2 x - 1$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$ or $\tan 2x = \frac{\sin 2x}{\cos 2x}$

$\cos 2x = 1 - 2\sin^2 x = 1 - 2\left[\frac{1}{4}\right]^2 = 1 - \cancel{2}\left[\frac{1}{\cancel{16}_8}\right]$

$= 1 - \frac{1}{8} = \frac{8}{8} - \frac{1}{8} = \frac{7}{8}$

$$\boxed{\cos 2x = \frac{7}{8}}$$

$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{-\sqrt{15}}{8} \div \frac{7}{8} = \frac{-\sqrt{15}}{\cancel{8}} * \frac{\cancel{8}}{7} = \frac{-\sqrt{15}}{7}$

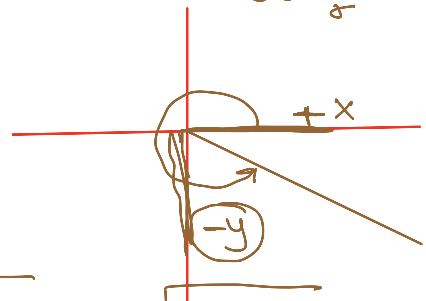
H.W
tan2x
بالقانون

8. $\sec x = 2$, x in Quadrant IV

مقلوب Cos

$\cos x = \frac{1}{2}$

$\cos = \frac{x}{r}$



$\sec x = \frac{2}{1} = \frac{r}{x}$

$r = 2 \qquad x = 1 \qquad y = \pm\sqrt{r^2 - x^2} = -\sqrt{r^2 - x^2}$

$= -\sqrt{2^2 - 1^2} = -\sqrt{4-1} = -\sqrt{3}$

$$\boxed{x = 1 \qquad y = -\sqrt{3} \qquad r = 2}$$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$ → $\sin 2x = \checkmark$

$= 1 - 2\sin^2 x$ → $\cos 2x = \checkmark$

$= 2\cos^2 x - 1$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$ or $\tan 2x = \frac{\sin 2x}{\cos 2x}$

$\sin 2x = 2\sin x \cos x$

$= 2\left[\frac{y}{r}\right]\left[\frac{x}{r}\right]$

$= \cancel{2}\left[\frac{-\sqrt{3}}{\cancel{2}}\right]\left[\frac{1}{2}\right] = \frac{-\sqrt{3}}{2}$

$$\boxed{\sin 2x = \frac{-\sqrt{3}}{2}}$$

$\cos 2x = 2\cos^2 x - 1 = 2\left[\frac{1}{2}\right]^2 - 1 = \cancel{2}\left[\frac{1}{2 \cdot \cancel{4}}\right] - 1$

$= \frac{1}{2} - \frac{2}{2} = \frac{-1}{2}$

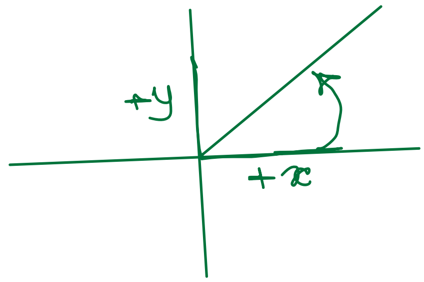
$$\boxed{\cos 2x = -1/2}$$

$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{-\sqrt{3}}{2} \div \frac{-1}{2} = -\frac{\sqrt{3}}{\cancel{2}} \times \frac{\cancel{2}}{1} = +\sqrt{3}$

10. $\cot x = \frac{2}{3}$, $\boxed{\sin x > 0}$ موجبة

معكوس

$\tan x = \pm\frac{3}{2} = \frac{y}{x}$



$\cot x = \frac{2}{3} = \frac{x}{y}$

$x = 2 \qquad y = 3 \qquad r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

$$\boxed{x = 2 \qquad y = 3 \qquad r = \sqrt{13}}$$

$\sin 2x = 2 \sin x \cos x = 2\left[\frac{y}{r}\right]\left[\frac{x}{r}\right] = 2\left[\frac{3}{\sqrt{13}}\right]\left[\frac{2}{\sqrt{13}}\right]$

$$\boxed{\sin 2x = \frac{12}{13}}$$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$ or $\tan 2x = \frac{\sin 2x}{\cos 2x}$

$\to \sin 2x = \checkmark$

$\to \cos 2x = \checkmark$

$\cos 2x = 2\cos^2 x - 1$

$= 2\left[\frac{x}{r}\right]^2 - 1$

$= 2\left[\frac{2}{\sqrt{13}}\right]^2 - 1 = 2\left[\frac{4}{13}\right] - 1 = \frac{8}{13} - \frac{13}{13}$

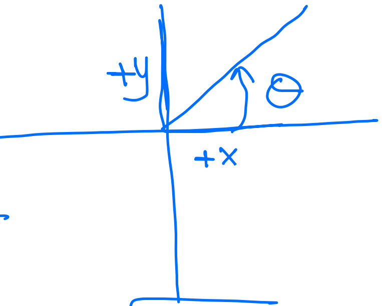
$$\boxed{\cos 2x = \frac{-5}{13}}$$

$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{12}{13} \div \frac{-5}{13} = \frac{12}{\cancel{13}} \times \frac{\cancel{13}}{-5} = -\frac{12}{5}$

$$\boxed{\tan 2x = \frac{-12}{5}}$$

51–54 . Evaluating an Expression Involving Trigonometric Functions Evaluate each expression under the given conditions.

54. $\tan 2\theta$; $\cos\theta = \frac{3}{5}$, θ in Quadrant I



$$\cos\theta = \frac{3}{5} = \frac{x}{r}$$

$x = 3$ $\quad r = 5$

$$y = \pm\sqrt{r^2 - x^2}$$
$$= +\sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16}$$
$$= 4$$

$$\boxed{x = 3 \qquad y = 4 \qquad r = 5}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)^2} = \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$$

$$= \frac{8/3}{1 - \frac{16}{9}} = \frac{8/3}{\frac{9}{9} - \frac{16}{9}} = \frac{8/3}{-7/9} = \frac{8}{3} \div \frac{-7}{9}$$

$$= \frac{8}{\cancel{3}_{1}} \times \frac{\cancel{9}^{3}}{-7} = \frac{24}{-7} = -\frac{24}{7}$$