1.7 Solving Inequalities

متباينات ...

$=$ → Equation

Greater than $>$ $\geq$

less than $<$ $\leq$

مجموعة الحل فترة

Objectives

- Solving Linear Inequalities
- Solving Nonlinear Inequalities
- Modeling with Inequalities

الخطية

الغيرخطية → Quadratic ∪ / rational

Solving Linear Inequalities

Some problems in algebra lead to **inequalities** instead of equations. An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols, $<$, $>$, $\leq$, or $\geq$. Here is an example of an inequality:

$$4x + 7 \leq 19$$

The table in the margin shows that some numbers satisfy the inequality and some numbers don't.

$4(1) + 7 = 11$

$4(2) + 7 = 15$

$4(4) + 7 = 23$

x	$4x + 7 \leq 19$
1	$11 \leq 19$ ✓
2	$15 \leq 19$ ✓
3	$19 \leq 19$ ✓
4	$23 \leq 19$ ✗
5	$27 \leq 19$ ✗

5–10 ■ Solutions? Let $S = \{-5, -1, 0, \frac{2}{3}, \frac{5}{6}, 1, \sqrt{5}, 3, 5\}$. Determine which elements of S satisfy the inequality.

10. $x^2 + 2 < 4$

$x = -5 \quad (-5)^2 + 2 = 27$

$x = -1 \quad (-1)^2 + 2 = 3$

$x = 0 \quad (0)^2 + 2 = 2$

$x = \frac{2}{3} \quad \left(\frac{2}{3}\right)^2 + 2 = 2\frac{4}{9}$

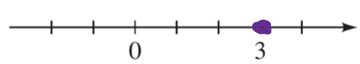
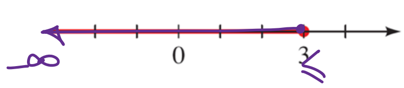
$x = \sqrt{5} \quad (\sqrt{5})^2 + 2 = 7$

x	$x^2 + 2 < 4$	
-5	$27 < 4$	✗
-1	$3 < 4$	✓
0	$2 < 4$	✓
$\frac{2}{3}$	$2\frac{4}{9} < 4$	✓
H.W $\frac{5}{6}$		
H.W 1		
$\sqrt{5}$	$7 < 4$	✗ ✗
H.W 3		
H.W 5		

To **solve** an inequality that contains a variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line.

The following illustration shows how an inequality differs from its corresponding equation:

$4x + 7 \leq 19$

	Solution	Graph
Equation: $4x + 7 = 19$	$\boxed{x = 3}$	
Inequality: $4x + 7 \leq 19$	$x \leq 3$	

$(-\infty, 3]$

To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign. These rules tell us when two inequalities are *equivalent* (the symbol $\Leftrightarrow$ means "is equivalent to").

In these rules the symbols A, B, and C stand for real numbers or algebraic expressions. Here we state the rules for inequalities involving the symbol ≤, but they apply to all four inequality symbols.

Rules for Inequalities

Rule	Description
1. $A \le B \quad \Leftrightarrow \quad A + C \le B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \le B \quad \Leftrightarrow \quad A - C \le B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \le B \quad \Leftrightarrow \quad CA \le CB$ لو رقم موجب	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \le B \quad \Leftrightarrow \quad CA \ge CB$ لو رقم سالب	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \le B \quad \Leftrightarrow \quad \frac{1}{A} \ge \frac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \le B$ and $C \le D$, then $A + C \le B + D$	Inequalities can be added.
7. If $A \le B$ and $B \le C$, then $A \le C$	Inequality is transitive.

عندضرب متباينه في اي رقم سالب يتم قلب اشارة المتباينه

Pay special attention to Rules 3 and 4. Rule 3 says that we can multiply (or divide) each side of an inequality by a *positive* number, but Rule 4 says that if we multiply each side of an inequality by a *negative* number, then we reverse the direction of the inequality.

For example, if we start with the inequality $3 < 5$

and multiply by 2, we get

$$6 < 10$$

but if we multiply by −2, we get

$$-6 > -10$$

An inequality is **linear** if each term is constant or a multiple of the variable.

To solve a linear inequality, we isolate the variable on one side of the inequality sign.

11–32 ■ Linear Inequalities Solve the linear inequality. Express the solution using interval notation and graph the solution set.

1 ✓ 2 فترات 3 أمثل المتباينة على خط الأعداد

15. $2 - 3x > 8$

1

Consider $2 - 3x = 8$

$-3x = 8 - 2$

$\dfrac{\cancel{-3}x}{\cancel{-3}} = \dfrac{6}{-3}$

$\boxed{x = -2}$

$x < -2$

2

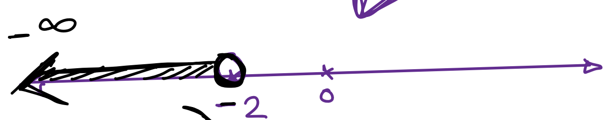
$2 - 3x > 8$ ✓

$-3x > 8 - 2$

$\dfrac{\cancel{-3}x}{\cancel{-3}} > \dfrac{6}{-3}$

$\boxed{x < -2}$

Graph



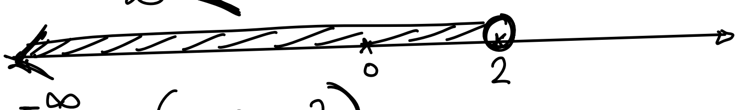
$(-\infty, -2)$

16. $1 < 5 - 2x$

$2x < 5 - 1$

$\dfrac{2x}{2} < \dfrac{4}{2}$

$x < 2$

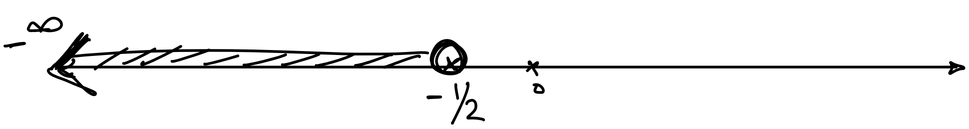


$(-\infty, 2)$

17. $2x + 1 < 0$

$\dfrac{2x}{2} < \dfrac{-1}{2}$

$x < -\frac{1}{2}$



$(-\infty, -\frac{1}{2})$

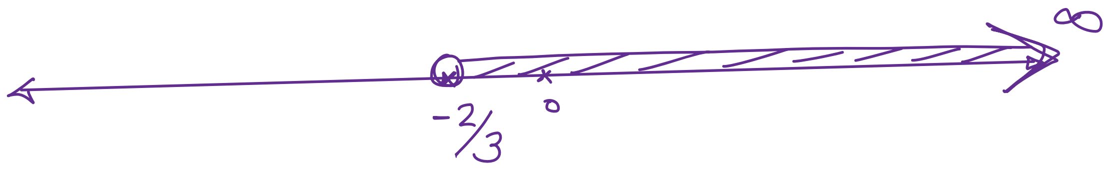
عند قسمة طرفي المتباينة على رقم
سالب يتم القلب

Solve the inequality $3x < 9x + 4$, and sketch the solution set.

$$3x - 9x < 4$$

$$\frac{-6x}{-6} < \frac{\cancel{4}^{\,2}}{-\cancel{6}_{\,3}}$$

$$x > -\frac{2}{3}$$



$$\left(-\frac{2}{3}, \infty\right)$$

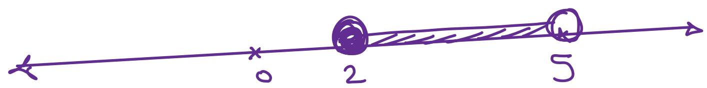
Solving a Pair of Simultaneous Inequalities

Solve the inequalities $4 \le 3x - 2 < 13.$

$4+2 \le 3x < 13+2$

$\frac{6}{3} \le \frac{\cancel{3}x}{\cancel{3}} < \frac{15}{3}$

$2 \le x < 5$



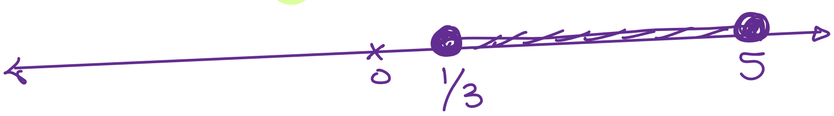
$[2, 5)$

27. $-6 \le 3x - 7 \le 8$

$-6+7 \le 3x \le 8+7$

$\frac{1}{3} \le \frac{\cancel{3}x}{\cancel{3}} \le \frac{15}{3}$

صغير $\frac{1}{3} \le x \le 5$ كبير



$\left[\frac{1}{3}, 5\right]$

Q **29.** $-2 < 8 - 2x \le -1$

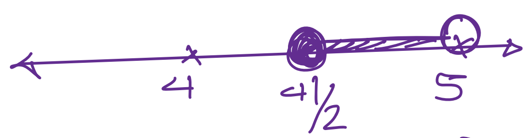
$<$ $<$

$-8-2 < -2x \le -1-8$

$\frac{-10}{-2} < \frac{\cancel{-2}x}{\cancel{-2}} \le \frac{-9}{-2}$

الكبير $5 > x \ge \frac{9}{2}$ الصغير

$\frac{9}{2} \le x < 5$



$[4\tfrac{1}{2}, 5)$

Solving a Quadratic Inequality

To solve inequalities involving squares and other powers of the variable, we use factoring, together with the following principle.

The Sign of a Product or Quotient

- If a product or a quotient has an *even* number of *negative* factors, then its value is *positive*.
- If a product or a quotient has an *odd* number of *negative* factors, then its value is *negative*.

$$\underbrace{ax^2+bx+c}_{(\ \ 1\ \ )(\ \ 2\ \ )} \overset{>}{<} 0$$

لابد ان يكون
الطرف الأيمن
صفر

Guidelines for Solving Nonlinear Inequalities

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use **test values** to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Use the sign table to find the intervals on which the inequality is satisfied. Check whether the **endpoints** of these intervals satisfy' the inequality. (This may happen if the inequality involves ≤ or ≥.)

The factoring technique that is described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol. If the inequality is not written in this form, first rewrite it, as indicated in Step 1.

Solve the inequality $x^2 + 5x + 6 > 0$ المهم

$6 \to 2, 3$

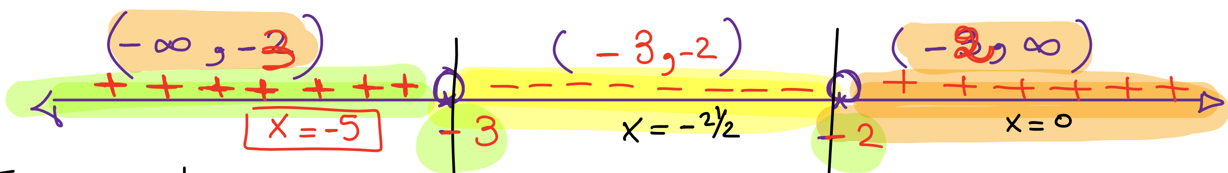
factor $x^2 + 5x + 6$

$(x+2)(x+3) = 0$

Critical value

$x + 2 = 0$ or $x + 3 = 0$

$x = -2$ $\qquad$ $x = -3$



Test	$(-\infty, -3)$	$(-3, -2)$	
$(x+2)(x+3)$	$(-5+2)(-5+3)$ $(-3)(-2)$ $+6$	$(-2\frac{1}{2}+2)(-2\frac{1}{2}+3)$ $-\frac{1}{2} * +\frac{1}{2}$ $-\frac{1}{4}$	$(0+2)(0+3)$ $2 * 3$ $+6$

$$S.S = (-\infty, -3) \cup (-2, \infty)$$

33–54 ■ Nonlinear Inequalities Solve the nonlinear inequality. Express the solution using interval notation, and graph the solution set.

$x^2 \le 5x - 6.$ اقل من (سالبة)

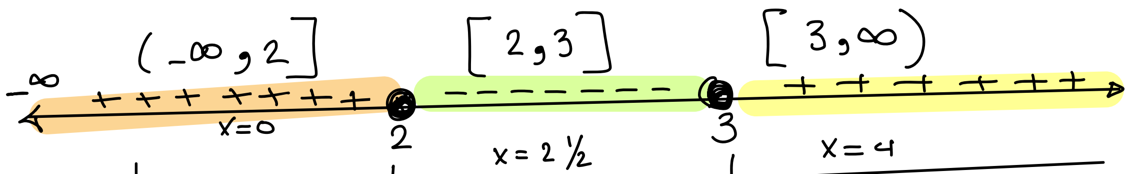
$x^2 - 5x + 6 \le 0$

$x^2 - 5x + 6$

$(x - 2)(x - 3) = 0$

$x - 2 = 0$ or $x - 3 = 0$

$x = 2$ $\quad$ $x = 3$ ← critical value



$(x-2)(x-3)$	$(0-2)(0-3)$ $-2 * -3$ $+6$	$(2\frac{1}{2}-2)(2\frac{1}{2}-3)$ $\frac{1}{2} * -\frac{1}{2}$ $\ominus$	$(4-2)(4-3)$ $+2$ $\quad$ $+1$

$S.S = [2, 3]$

40. $x^2 < x + 2$

انقل بس

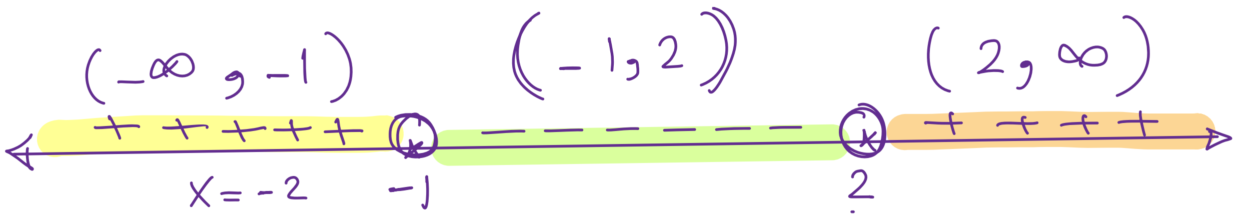
$x^2 - x - 2 < 0$

$x^2 - x - 2$

$(x + 1)(x - 2) = 0$

$x + 1 = 0$ or $x - 2 = 0$

$x = -1$ $\quad$ $x = 2$ ← critical value



Test	$X = -2$	$X = 0$	$X = 3$
$(x+1)(x-2)$	$(-2+1)(-2-2)$ $-1 * -4$ $+4$	$(0+1)(0-2)$ $+1 * -2$ $-$	$(3+1)(3-2)$ $4 * 1$ 4

$S.S = (-1, 2)$

Solve the inequality $2x^2 - x > 1$.

$2x^2 - x - 1 > 0$

$2x^2 - 1x - 1 = 0$

$(2x^2 - 2x) + (x - 1) = 0$

$2x(x-1) + (x-1) = 0$

$(x-1)(2x+1) = 0$

$x - 1 = 0$ or $2x + 1 = 0$

$x = 1$

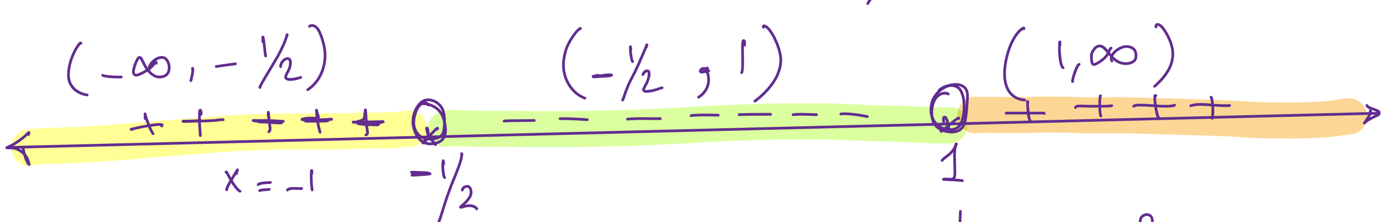
$\frac{2x}{2} = \frac{-1}{2}$

$x = -\frac{1}{2}$ critical value

$2 * -1 = -2$

$\boxed{-2 * 1}$

$-2 + 1 = -1$



	$x = -1$	$x = 0$	$x = 2$
$(x-1)$ $(2x+1)$	$(-1-1)$ $(2*-1+1)$ $-2+1$ $= -2 * -1$ $+2$	$(0-1)$ $(2*\cancel{0}+1)$ $= -1 * +1$ $= -1$	$(2-1)(2 \cdot 2 + 1)$ $1 * 5$ $+5$

$S.S = (-\infty, -\frac{1}{2}) \cup (1, \infty)$

Solve the inequality $x(x-1)^2(x-3) < 0$.

Critical value.

$x = 0$ $\quad$ $(x-1)^2 = 0$ $\quad$ $x - 3 = 0$

$x - 1 = 0$ $\quad$ $x = 3$

$x = 0$

$x = 1$

	$(-\infty, 0)$ $x=-1$	$(0, 1)$ $x=\frac{1}{2}$	$(1, 3)$ $x=2$	$(3, \infty)$ $x=4$
x	$(-1) = -$	$\left(\frac{1}{2}\right) = +$	(2) $+$	(4) $+$
$(x-1)^2$	$(-1-1)^2 = +$	$\left(\frac{1}{2}-1\right)^2 = +$	$(2-1)^2$ $+$	$(4-1)^2$ $+$
$(x-3)$	$(-1-3) = -$	$\left(\frac{1}{2}-3\right) = -$	$(2-3)$ $-$	$(4-3)$ $+$
	$+ + + + + + + + +$	$- - - - - -$	$- - - - - - -$	$+ + + +$

0 $\quad$ 3

$$S.S = (0, 1) \cup (1, 3)$$

H.W

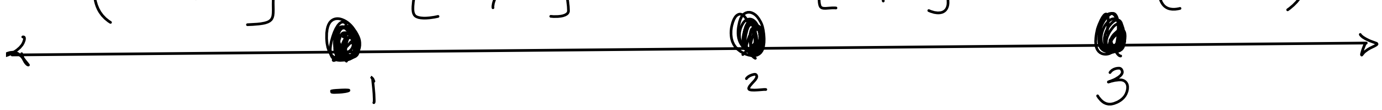
51. $(x-2)^2(x-3)(x+1) \le 0$

Critical value

$(x-2)^2=0$ $\quad$ $x-3=0$ $\quad$ $x+1=0$

$x=2$ $\quad$ $x=3$ $\quad$ $x=-1$

$(-\infty,-1]$ $\quad$ $[-1,2]$ $\quad$ $[2,3]$ $\quad$ $[3,\infty)$



72. $x^5 > x^2$

$$x^5 - x^2 > 0$$

$$x^5 - x^2 = 0$$

$$x^2\left(x^3 - 1\right) = 0$$

فرق بين مكعبين

Challenge

■ **Solving an Inequality Involving a Quotient**

Solve the inequality $\frac{1+x}{1-x} \geq 1.$

① لابد ان يكون الطرف الايمن صفر

② اقارن كسر واحد بالصفر
الطرف الايسر كسر واحد
ومن حاله وجود كسرين لية
توحيد المقامات

الحل

$$\frac{1+x}{1-x} - 1 \geq 0$$

$$\frac{1+x}{1-x} - \frac{1\,(1-x)}{(1-x)} \geq 0$$

$$\frac{\cancel{1}+x-\cancel{1}+x}{1-x} \geq 0$$

$$\boxed{\frac{2x}{1-x} \geq 0}$$

كسر اقارنه بالصفر

Critical value

(المقام) = صفر

$1-x=0$

$\boxed{1=x}$

اصفار (المقام)

الدائرة مفتوحة

البسط = صفر

$2x = 0$

$\frac{2x}{2} = \frac{0}{2}$

$x = 0$

لانه ① لعتبر (مقام) من اصفار المقام

$(-\infty, 0]$ $[0, 1)$ $(1, \infty)$

$- - - - - \quad 0 \quad + + + + \quad 1 \quad - - - - - -$

$x=-1$ $x=\frac{1}{2}$ $x=2$

$$\frac{2x}{1-x}$$

$$\frac{2*-1}{1--1} = \frac{-2}{2} \quad \ominus$$

$$\frac{2*\frac{1}{2}}{1-\frac{1}{2}} = \frac{+}{+}$$

$$\frac{2*2}{1-2} = \frac{+}{-}$$

$$S.S = [0, 1)$$

55–72 ■ Inequalities Involving Quotients Solve the nonlinear inequality. Express the solution using interval notation, and graph the solution set.

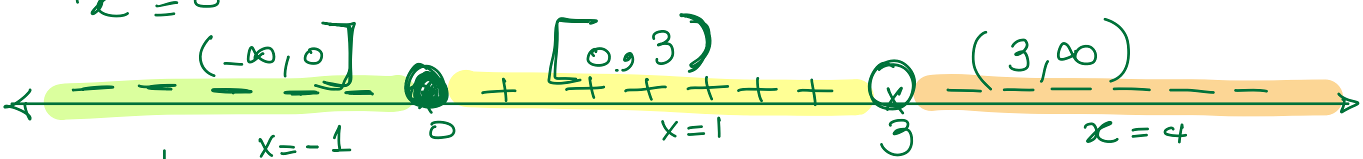
60. $\dfrac{3+x}{3-x} \geq 1$

$$\frac{3+x}{3-x} - \frac{1\,(3-x)}{(3-x)} \geq 0 \longrightarrow \frac{\cancel{3} + x - \cancel{3} + x}{3-x} \geq 0$$

$$\frac{2x}{3-x} \geq 0$$

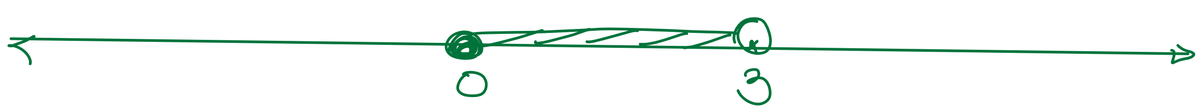
$$\frac{2x}{2} = \frac{0}{2} \qquad\qquad 3 - x = 0$$

$$x = 0 \qquad\qquad 3 = x$$



$\dfrac{2x}{3-x}$	$\dfrac{2*-1}{3--1} = \dfrac{-}{+}$	$\dfrac{2*1}{3-1} = \dfrac{+}{+}$	$\dfrac{2*4}{3-4} = \dfrac{+}{-}$

S.S $[0, 3)$



65. $\frac{6}{x-1} - \frac{6}{x} \geq 1$

LCD = $x(x$

$$\frac{6}{x-1} - \frac{6}{x} - 1 \geq 0$$

$$\frac{6 \cdot x}{(x-1)\,x} - \frac{6\,(x-1)}{x\,(x-1)} - \frac{1 * x\,(x-1)}{* \, x\,(x-1)} \geq 0$$

$-x$

$$\frac{\cancel{6x} - \cancel{6x} + 6 - x^2 + x}{x\,(x-1)} \geq 0$$

$$\frac{-x^2 + x + 6}{x\,(x-1)} \geq 0 \qquad * -1$$

$$\frac{x^2 - x - 6}{x\,(x-1)} \leq 0 \longrightarrow \frac{(x+2)(x-3)}{x\,(x-1)} \leq 0$$

سالب

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x + 2 = 0 \qquad x - 3 = 0$

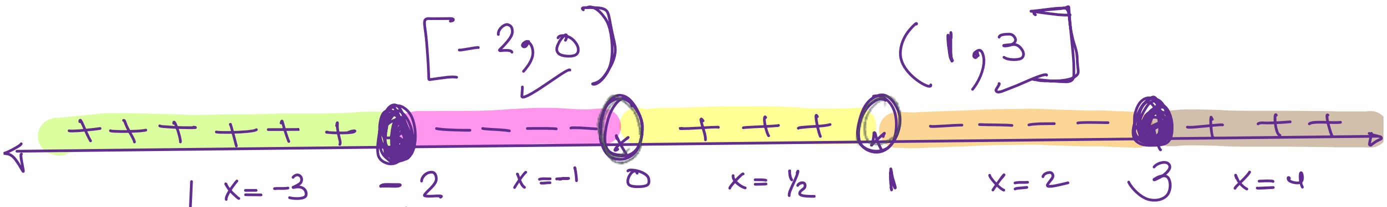
$\boxed{x = -2} \qquad \boxed{x = 3}$

$x\,(x-1) = 0$

$x = 0$ or $x - 1 = 0$

$\boxed{x = 0} \qquad \boxed{x = 1}$

اهمأ ، (خطأ)



$\frac{(x+2)(x-3)}{x(x-1)}$	$x=-3$	$x=-1$	$x=\frac{1}{2}$	$x=2$	$x=4$
	$\frac{(-3+2)(-3-3)}{-3(-3-1)}$	$\frac{(-1+2)(-1-3)}{-1(-1-1)}$	$\frac{(\frac{1}{2}+2)(\frac{1}{2}-3)}{\frac{1}{2}(\frac{1}{2}-1)}$	$\frac{(2+2)(2-3)}{2(2-1)}$	$\frac{(4+2)(4-3)}{4(4-1)}$
	$\frac{- * -}{- * -}$	$\frac{+ * -}{- * -}$	$\frac{+ * -}{+ * -}$	$\frac{+ * -}{+ \times +}$	$\frac{+ * +}{+ * +}$
	$+$	$\frac{-}{+}$ $-$	$\frac{-}{-}$	$\frac{-}{+}$ $-$	$+$

$$S.S = [-2, 0) \cup (1, 3]$$