CHAPTER 10
LIMITS & CONTINUITY

النهايات الاتصال

Chapter Objectives

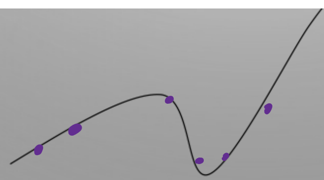
- To study limits and their basic properties.
- To study one-sided limits, infinite limits, and limits at infinity.
- To study continuity and to find points of discontinuity for a function.
- To develop techniques for solving nonlinear inequalities.

10.1) Limits

WHAT IS A LIMIT?

- UP TO NOW YOU HAVE USED ALGEBRA AND GEOMETRY TO SOLVE PROBLEMS WHERE THINGS ARE BASICALLY STAYING THE SAME OR CHANGING AT THE SAME RATE.
- CALCULUS IS A BRANCH OF MATHEMATICS THAT DEALS WITH THINGS THAT ARE CHANGING.

Volume of an expanding balloon



The rate of change at any point on a curve

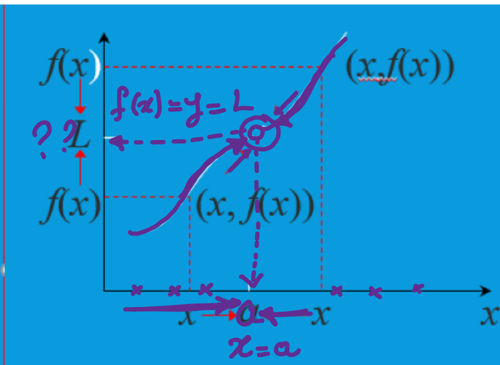
Acceleration of rocket ship that is changing every part of a second

WHAT IS A LIMIT?

- ONE OF THE MOST BASIC AND FUNDAMENTAL IDEAS OF CALCULUS IS LIMITS.
- LIMITS ALLOW US TO LOOK AT WHAT HAPPENS IN A **VERY, VERY** SMALL REGION AROUND A POINT.
- TWO OF THE MAJOR FORMAL DEFINITIONS OF CALCULUS DEPEND ON LIMITS.

DEFINITION OF LIMIT OF A FUNCTION

- Suppose that the function $f(x)$ is defined for all values of x near a, but not necessarily at a. If as x approaches a (without actually attaining the value a), $f(x)$ approaches the number L, then we say that L is the limit of $f(x)$ as x approaches a, and write



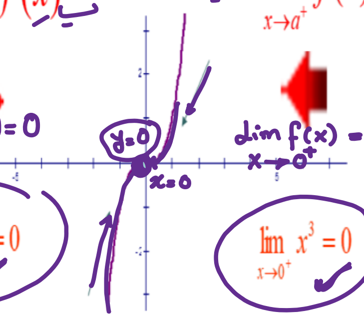
No matter how x approaches a, $f(x)$ approaches L.

LEFT & RIGHT HAND LIMITS

$f(x) = x^3$

$\lim_{x\to a^-} f(x)$ $\qquad$ $\lim_{x\to a^+} f(x)$

$\lim_{x\to 0^-} f(x) = 0$ $\qquad$ $\lim_{x\to 0^+} f(x) = 0$



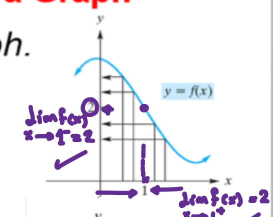
$\lim_{x\to 0^-} x^3 = 0$ ✓ $\qquad$ $\lim_{x\to 0^+} x^3 = 0$ ✓

- The limit of $f(x)$ as x approaches a is the number L written as
$$\lim_{x\to a} f(x) = L$$

Example 1 – Estimating a Limit from a Graph

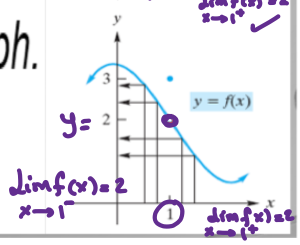
a. *Estimate* $\lim_{x\to 1} f(x)$ *from the graph.*

Solution: $\lim_{x\to 1} f(x) = 2$



b. Estimate $\lim_{x\to 1} f(x)$ from the graph.

Solution: $\lim_{x\to 1} f(x) = 2$



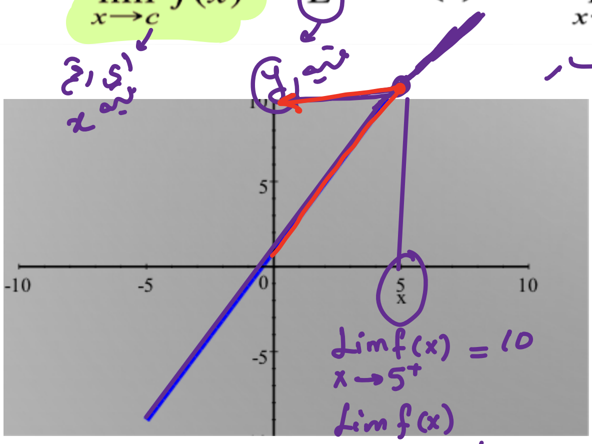
$\lim_{x \to 0} f(x) = 0$

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

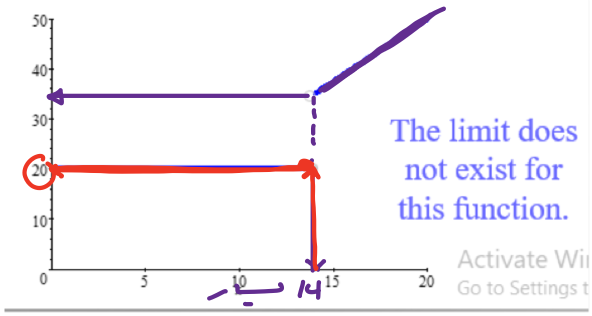
$y = L$ $\quad$ $y = L$



$\lim_{x \to 5^+} f(x) = 10$

$\lim_{x \to 5^-} f(x) = 10$

$\lim_{x \to 5} f(x) = \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 10$



$\lim_{x \to 14} f(x) = ??$

$\lim_{x \to 14^+} f(x) = 35 \neq \lim_{x \to 14^-} f(x) = 20$

lim does not exist

PROBLEM 10.1

In Problems 1–4, use the graph of f to estimate each limit, if it exists.

1. Graph of f appears in Figure 10.10.

(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 1} f(x) = 0$ (c) $\lim_{x \to 2} f(x) = 1$

$\lim_{x \to 0^+} = 1$

$\lim_{x \to 0^-} = 1$

$\lim_{x \to 0} f(x) = 1$

$\lim_{x \to 1^+} = 0$

$\lim_{x \to 1^-} = 0$

$\lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2^-} f(x) = 1$

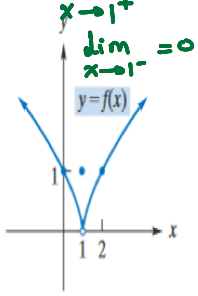


FIGURE 10.10

3. Graph of f appears in Figure 10.12.

(a) $\lim_{x \to -1} f(x) = 1$ (b) $\lim_{x \to 1} f(x)$ does not exist (c) $\lim_{x \to 2} f(x) = 3$

$\lim_{x \to -1^+} f(x) = 1$

$\lim_{x \to -1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 2$

$\lim_{x \to 1^-} = 1$

$\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2^-} f(x) = 3$

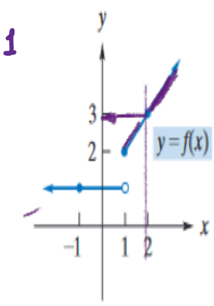


FIGURE 10.12

4. Graph of f appears in Figure 10.13.

(a) $\lim_{x \to -1} f(x) = -1$ (b) $\lim_{x \to 0} f(x)$ does not exist (c) $\lim_{x \to 1} f(x) = 1$

$\lim_{x \to -1^+} f(x) = -1$

$\lim_{x \to -1^-} f(x) = -1$

$\lim_{x \to 0^+} f(x) = +\infty$

$\lim_{x \to 0^-} f(x) = -\infty$

$\neq$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1^-} f(x) = 1$

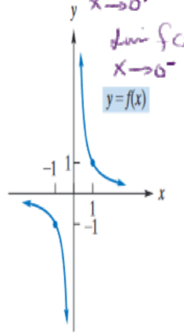


FIGURE 10.13

Properties of Limits

حفظ مهم

1. $\lim_{x\to a} f(x) = \lim_{x\to a} c = c$ where c is a constant $\longrightarrow \lim_{x\to 3} 7 = 7$ $\quad 4 \lim_{x\to 3} \pi = \pi$

خويّة اي دالة ثابتة = نفس الدالة $\quad \lim_{x\to -4} e^4 = e^4$

2. $\lim_{x\to a} x^n = a^n$ for any positive integer n

$\lim_{x\to 2} x^4 = 2^4 = \checkmark$ $\quad 4 \lim_{x\to 4} x^{-1/2} = x^{-1/2}$

3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

$\lim_{x\to 0}[x^2 + 3x - 5] = \lim_{x\to 0} x^2 + \lim_{x\to 0} 3x - \lim_{x\to 0} 5$

4. $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

$\lim_{x\to 3} x^2 * 2x = \lim_{x\to 3} x^2 * \lim_{x\to 3} 2x$

5. $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x) \longrightarrow \lim_{x\to 0} \frac{1}{2} x^2 = \frac{1}{2} \lim_{x\to 0} x^2$

6. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$ $\quad \lim_{x\to 0} \frac{3x^2}{2x-5} = \frac{\lim_{x\to 0} 3x^2}{\lim_{x\to 0}(2x-5)}$

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ $\quad = \lim_{x\to 4} \sqrt[5]{5x^2-5}$

$= \sqrt[5]{\lim_{x\to 4}(5x^2-5)}$

Example 3 –

نهاية اى دالة ثابتة = نفسها

$\lim_{x \to 2} 7 = 7$

$\lim_{x \to -5} 7 = 7$

$\lim_{x \to 6} x^2 = (6)^2 = 36$

$\lim_{t \to 2} t^4 = (2)^4 = 16$

Limit of a Polynomial Function

Find an expression for the polynomial function,

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$$

$f(x) = 3x^5 - 2x^4 + \frac{1}{10} x^2 - 3x + 5$

يتم التعويض مباشرة عن كل x بقيمة النهاية

Solution:

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 \right)$$

$$= c_n \lim_{x \to a} x^n + c_{n-1} \lim_{x \to a} x^{n-1} + \ldots + c_1 \lim_{x \to a} + \lim_{x \to a} c_0$$

$$= c_n a^n + c_{n-1} a^{n-1} + \ldots + c_1 a + c_0$$

$$= f(a)$$

where $\lim_{x \to a} f(x) = f(a)$

13. $\lim_{x \to -2} (3x^3 - 4x^2 + 2x - 3)$

$3(-2)^3 - 4(-2)^2 + 2(-2) - 3$

$3(-8) - 4(4) - 4 - 3$

$-24 - 16 - 4 - 3 = -47$

15. $\lim_{t\to -3} \dfrac{t-2}{t+5} = \dfrac{(-3)-2}{(-3)+5} = \dfrac{-5}{2}$

تحليل للبسط

6. $\lim_{x\to -3} \dfrac{x^2-9}{x+3}$

$\dfrac{(\quad)\cancel{(\quad)}}{\cancel{(\quad)}}$

$= \dfrac{(-3)^2-9}{(-3)+3} = \dfrac{9-9}{0} = \dfrac{0}{0}$

$\lim_{x\to -3} \dfrac{x^2-3^2}{x+3} = \lim_{x\to -3} \dfrac{\cancel{(x+3)}(x-3)}{\cancel{x+3}}$

$= \lim_{x\to -3} x-3 = (-3)-3 = -6$

\# $\lim_{x\to -3} \dfrac{x^2-9}{x+3}$

$-3.1 \quad -3.01 \quad -3.001 \quad -3$

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-6.1	-6.01	-6.001 $\boxed{-6}$	-5.999	-5.99	-5.9

$f(x) = f(-3)$

$f(-3.1) = -6.1$ $\qquad$ $f(-2.9) = -5.9$

$f(-3.01) = -6.01$ $\qquad$ $f(-2.99) = -5.99$

$f(-3.001) = -6.001$ $\qquad$ $f(-2.999) = -5.999$

estimate of limit: -6

[illegible]

21. $\lim_{x \to -2} \dfrac{x^2 + 2x}{x + 2}$

عامل مشترك

$x \quad x$

$$\frac{(-2)^2 + 2(-2)}{(-2) + 2} = \frac{4 - 4}{0} = \frac{0}{0}$$

$$\lim_{x \to -2} \frac{x\cancel{(x+2)}}{\cancel{x+2}} = \lim_{x \to -2} x = -2$$