3.1 Quadratic Functions and Models

Objectives

$ax^2 + bx + c = 0$

$f(x) = ax^2 + bx + c$

$a \neq 0$

- Graphing Quadratic Functions Using the Standard Form
- Maximum and Minimum Values of Quadratic Functions
- Modelling with Quadratic Function

QUADRATIC FUNCTIONS

A **quadratic function** is a polynomial function of degree 2. So a quadratic function is a function of the form

$$f(x) = ax^2 + bx + c \qquad a \neq 0$$

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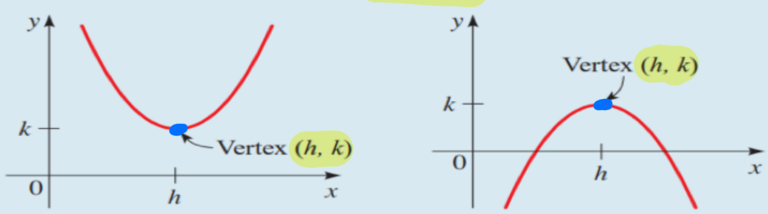
STANDARD FORM OF A QUADRATIC FUNCTION

A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

عكس الاشارة   نفس

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with **vertex** (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.



$f(x) = a(x - h)^2 + k, \ a > 0$ $f(x) = a(x - h)^2 + k, \ a < 0$

parabola open up   parabola open down

$f(x) = 2(x - 2)^2 + 3$ $a = 2 > 0 \to \cup$

$h = +2$ عكس الاشارة $K = 3$ نفس الاشارة

$f(x) = -4(x + 2)^2 - 3$ $a = -4 < 0 \to \cap$

$h = -2$ $K = -3$

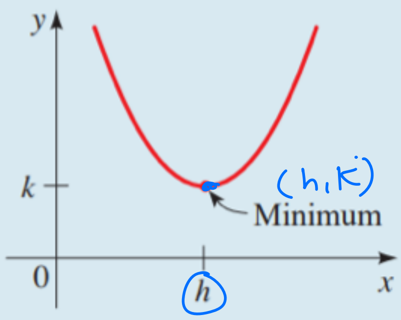
Vertex عكسها   نفسها

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC FUNCTION

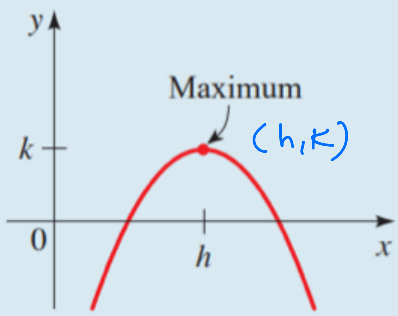
Let f be a quadratic function with standard form $f(x) = a(x - h)^2 + k$. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$, then the **minimum value** of f is $f(h) = k$.

If $a < 0$, then the **maximum value** of f is $f(h) = k$.



$f(x) = a(x - h)^2 + k, a > 0$



$f(x) = a(x - h)^2 + k, a < 0$

EXAMPLE 2 ■ Minimum Value of a Quadratic Function

Consider the quadratic function $f(x) = 5x^2 - 30x + 49$.

(a) Express f in standard form.

(b) Sketch a graph of f.

(c) Find the minimum value of f.

a) $f(x) = 5x^2 - 30x + 49 \longrightarrow a(x-h)^2 + K$

آكمال المربع

$= 5\left[\underbrace{x^2 - 6x}_{\text{آكمال مربع}}\quad\right] + 49$

$\left(\frac{-6}{2}\right)^2 = 9$

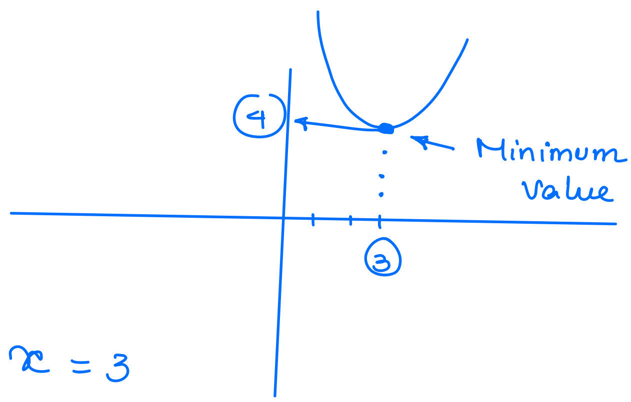
$= 5\left[x^2 - 6x + 9\right] + 49 - 45$

$5 * 9 = 45$

$= 5(x-3)^2 + 4 \longrightarrow a(x-h)^2 + K$

$a = 5 > 0$ parabola open up $\quad h = 3 \quad K = 4$

b) Sketch



c) Min value is 4 when $x = 3$

EXAMPLE 3 ■ Maximum Value of a Quadratic Function

Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express f in standard form.

(b) Sketch a graph of f.

(c) Find the maximum value of f.

a) $f(x) = -x^2 + x + 2 \longrightarrow a(x-h)^2 + k$

$$= -\underbrace{[x^2 - x}_{\text{آكمال طرح}}] + 2 \qquad \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$$

$$= -\left[x^2 - x + \frac{1}{4}\right] + 2 + \frac{1}{4}$$

$$-\frac{1}{4}$$

$$2 * \frac{4}{4} = \frac{8}{4} + \frac{1}{4} = \frac{9}{4}$$

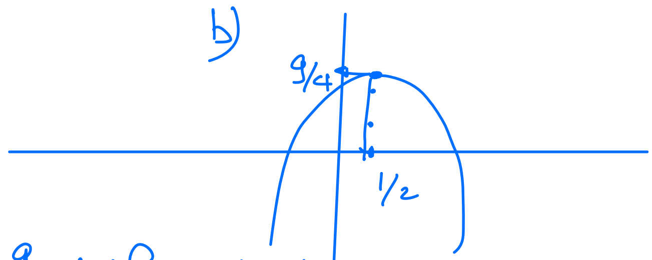
$$= -1\left(x - \frac{1}{2}\right)^2 + \frac{9}{4} \longrightarrow a(x-h)^2 + k$$

$a = -1 < 0$ parabola open down

$h = \frac{1}{2}$

$K = \frac{9}{4}$

b)



c) Maximum value at $\frac{9}{4}$ when $x = \frac{1}{2}$

25–34 ■ Maximum and Minimum Values A quadratic function f is given. **(a)** Express f in standard form. **(b)** Sketch a graph of f. **(c)** Find the maximum or minimum value of f.

29. $f(x) = -x^2 - 3x + 3 \longrightarrow a(x-h)^2 + K$

$$= -(\underbrace{x^2 + 3x}_{\text{اكمال تربيع}}\quad) + 3$$

$$= -\left(x^2 + 3x + \tfrac{9}{4}\right) \quad + \tfrac{4\times 3}{4} + \tfrac{9}{4}$$

$-9/4$

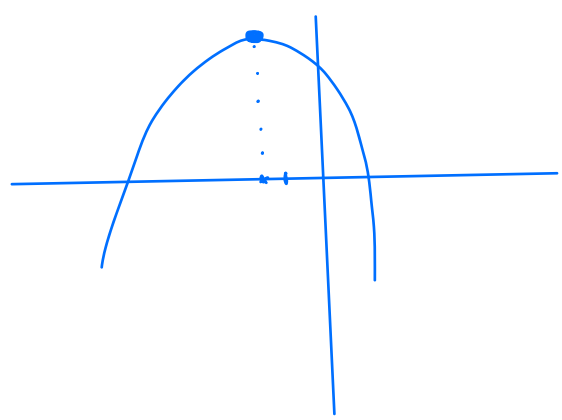
$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$

$12 + 9 = 21$

$$= -\left(x + \tfrac{3}{2}\right)^2 + \tfrac{21}{4} \longrightarrow a(x-h)^2 + K$$

$a = -1 < 0$ parabola open down

$h = -\frac{3}{2}$ $K = \frac{21}{4}$



c) It has a maximum value of $\frac{21}{4}$ at $x = -\frac{3}{2}$

31. $g(x) = 3x^2 - 12x + 13$ $\longrightarrow$ $a(x-h)^2 + K$

$g(x) = 3\left[\underbrace{x^2 - 4x \qquad}_{\text{[illegible]}}\right] + 13$

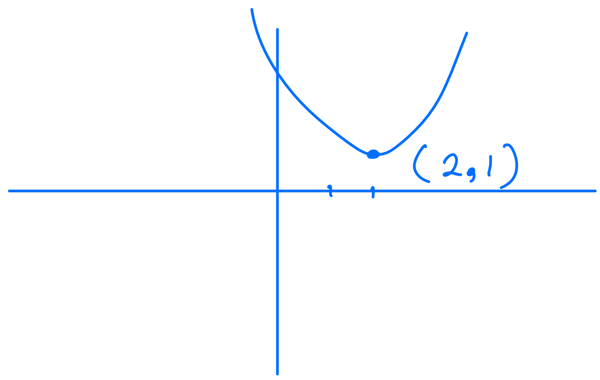
$= 3\left[x^2 - 4x + 4\right] + 13 - 12$ $\qquad \left(\frac{-4}{2}\right)^2 = 4$

$3 * 4 = 12$

$= 3(x-2)^2 + 1$

$a = 3 > 0$ parabola open up $\qquad h = 2 \qquad K = 1$

b)



$g(x)$ has a minimum point of 1 at $x = 2$

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC FUNCTION

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

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$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

$y = ax^2 + bx + c$

If $a < 0$, then the **maximum value** is $f\left(-\frac{b}{2a}\right)$.

$y = ax^2 + bx + c$

$\frac{-b}{2a}$

EXAMPLE 4 ■ Finding Maximum and Minimum Values of Quadratic Functions

Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$

(b) $g(x) = -2x^2 + 4x - 5$

a) $f(x) = x^2 + 4x$

$a = 1 > 0 \; \cup \qquad b = 4$

$x = \frac{-b}{2a} = \frac{-4}{2*1} = -2$

$f(x) = f\left(\frac{-b}{2a}\right) = f(-2) = (-2)^2 + 4(-2)$

$= 4 - 8 = -4$

Vertex $(-2, -4)$

$a = 1 > 0 \longrightarrow \cup \rightarrow$ It has a minimum point of -4 at $x = -2$

(b) $g(x) = -2x^2 + 4x - 5$

$a = -2 \qquad b = 4$

$x = \frac{-b}{2a} = \frac{-4}{2(-2)} = +1$

$f(x) = f\left(\frac{-b}{2a}\right) = f(1) = -2(1)^2 + 4(1) - 5 = -3$

Vertex $(1, -3) \longrightarrow a = -2 < 0 \; \cap$

The parabola has a maximum point of -3 at $x = 1$

$(1, -3)$

35–44 ■ Formula for Maximum and Minimum Values

Find the maximum or minimum value of the function.

36. $f(x) = 3 - 4x - x^2$

$$f(x) = -x^2 - 4x + 3$$

$a = -1 \qquad b = -4$

$$x = \frac{-b}{2a} = \frac{-(-4)}{2(-1)} = \frac{4}{-2} = -2$$

$$f(x) = f\left(\frac{-b}{2a}\right) = f(-2) = -(-2)^2 - 4(-2) + 3$$

$$= -4 + 8 + 3 = 7$$

$a = -1 < 0$ parabola open down. It has a maximum point $(-2, 7)$

38. $f(x) = 6x^2 - 24x - 100$

$a = 6 \qquad b = -24$

$x = \frac{-b}{2a} = \frac{-(-24)}{2 * 6} = \frac{24}{12} = 2$

$f(x) = f\left(\frac{-b}{2a}\right) = f(2) = 6(2)^2 - 24(2) - 100$

$= 24 - 48 - 100$

$= -124$

$a = 6 > 0$ ∪ It has a minimum point

$(2, -124)$

41. $h(x) = \frac{1}{2}x^2 + 2x - 6$

$a = \frac{1}{2}$ $\qquad b = 2$

$x = \frac{-b}{2a} = \frac{-2}{\cancel{2} * \frac{1}{\cancel{2}}} = -\frac{2}{1} = -2$

$f(x) = f\left(\frac{-b}{2a}\right) = f(-2) = \frac{1}{2}(-2)^2 + 2(-2) - 6$

$= \frac{1}{\cancel{2}_{1}}(\cancel{4}^{2}) - 4 - 6$

$2 \quad -10 = -8$

$a = \frac{1}{2} > 0$ ∪ It has minumum point

$(-2, -8)$