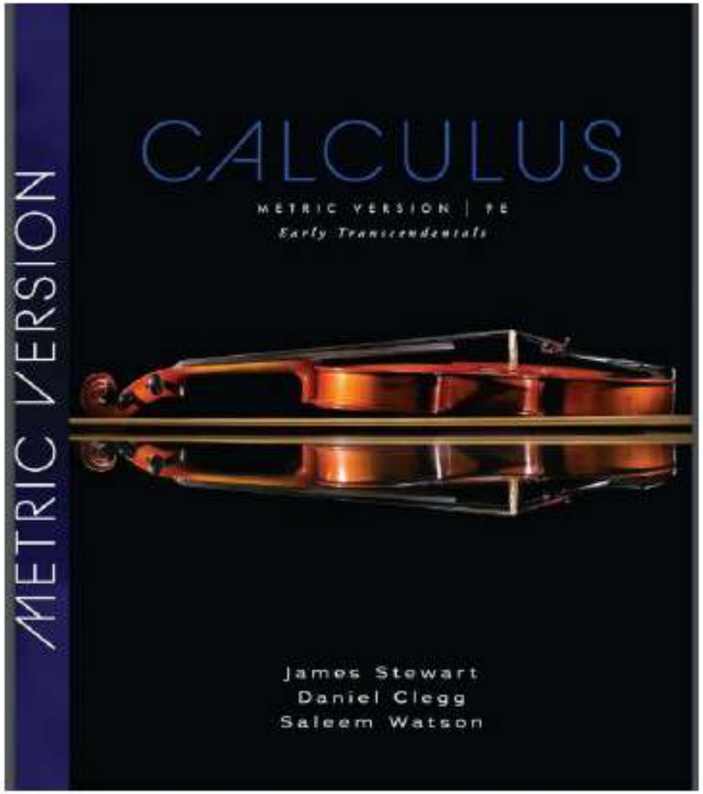
Calculus I



This helping matiral is taken from Stewart, J., Clegg, D.K., & Watson, S. (2021). Calculus: Early Transcendentals (9th ed.).

- Limits
- differentiation
- Integration

2 Limits and Derivatives

النهايات　　　الاشتقاق

خط التماس　　　السرعة اللحظية

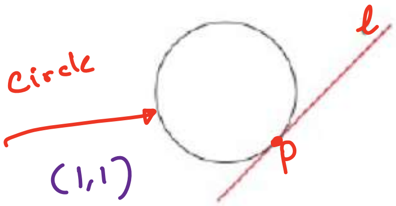
2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

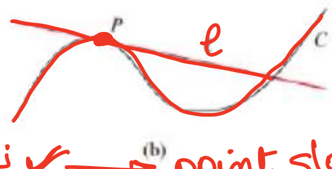
The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." We can think of a tangent to a curve as a line that touches the curve and follows the same direction as the curve at the point of contact. How can this idea be made precise?

لمس



(a)



(b)

$(1,1)$

$$\boxed{y - y_1 = m(x - x_1)}$$

نقطة + ميل → point slope form

$(1,1)$

نقطة التقاطع مع محور السينات x xx

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

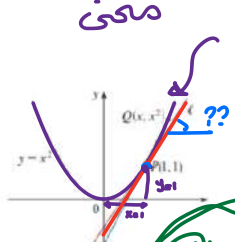
منحنى

نقطة واقعة على كل من الخط المستقيم والمنحنى

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

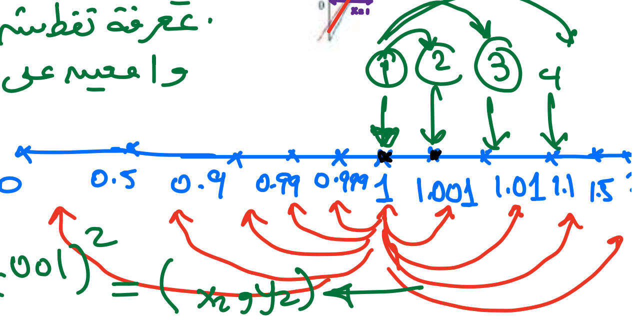
$$m_{\tan} = \frac{y_2 - y_1}{x_2 - x_1}$$

معادلة ميل اي خط مستقيم تعرفة نقطتين واحدة على الخط



x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

1st point $(1,1)$



$x = 1.001 \longrightarrow y = ??$

$y = x^2 = (1.001)^2 = (x_2, y_2)$

$y = x^2$ محادثة منحنى

$\leftarrow (x_2, y_2)$ افرض نقطة

$x_2 = x \longrightarrow y_2 = x^2 \longrightarrow$ point (x, x^2)

النقطة الثانية

Point ①

Given $(1, 1)$

نقطة التماس

Point ②

$(x, x^2) \longrightarrow$ $y = x^2$, $x_2 = x$, $y_2 = x^2$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2 - 1}{x - 1}$$

الصورة العامة لميل الخط المستقيم الـ curve $y = x^2$ at any point

> 1

x (افرض)	m_{PQ} (ميل) $= \frac{x^2-1}{x-1}$
2	3 $\rightarrow x=2 \rightarrow \frac{2^2-1}{2-1} = 3$
1.5	2.5 $\rightarrow x=1.5 \rightarrow m = \frac{1.5^2-1}{1.5-1} = 2.5$
1.1	2.1 $\rightarrow x=1.1 \rightarrow m = \frac{1.1^2-1}{1.1-1} = 2.1$
1.01	2.01
1.001	2.001

< 1 $\quad m = \frac{x^2-1}{x-1}$

x	m_{PQ}
0	1 $\rightarrow x=0 \rightarrow m = \frac{0^2-1}{0-1} = 1$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

$x = 1$

x	0	0.5	0.9	0.99	0.999	1.001	1.01	1.1	1.5	2
m	1	1.5	1.9	1.99	1.999	2.001	2.01	2.1	2.5	3

$\boxed{2}$

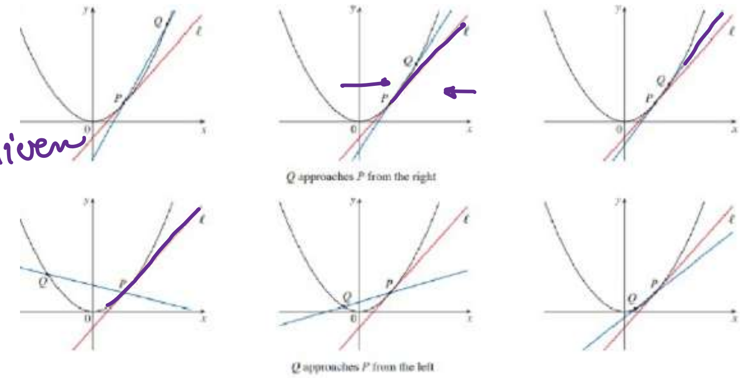
كلما اقتربت الـ x من $\boxed{1}$ (نقطة التماس) $\leftarrow$ اقترب الميل من $\boxed{2}$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through (1, 1) as

$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (1, 1) \rightarrow$ Given

$m \simeq 2$



Q approaches P from the right

Q approaches P from the left

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$y = 2x - 2 + 1$

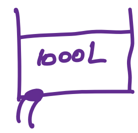
$\boxed{y = 2x - 1}$ ← معادلة الخط المستقيم المماس للمنحنى

$y = x^2$ at point $(1, 1)$

$\boxed{m = 2}$ وذلك لمعرفة القيمة التقريبية للميل

Exercise 2.1

1. A tank holds 1000 liters of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in liters) after t minutes.

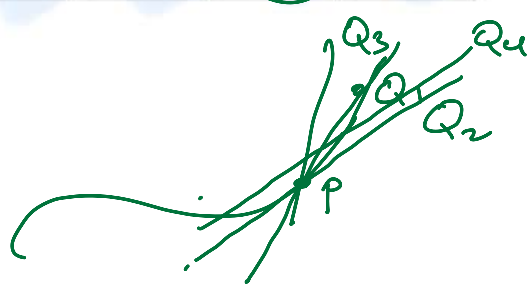


t (min)	5	10	15	20	25	30
V (L)	694	444	250	111	28	0

x y حجم الماء المتبقي في التانك

t_1 t_2 P v_2

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

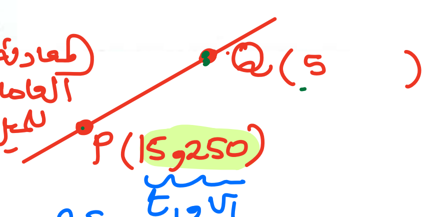


(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25,$ and 30.

$t \to x$ $V \to y$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{v_2 - v_1}{t_2 - t_1}$$

المعادلة العامة للميل



$Q_1 \to t_2 = 5 \longrightarrow V_2 = 694 \longrightarrow m = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{694 - 250}{5 - 15} = -44.4$ l_1

$Q_2 \to t_2 = 10 \longrightarrow V_2 = 444 \longrightarrow m = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{444 - 250}{10 - 15} = -38.8$ l_2

$Q_3 \to t_2 = 20 \longrightarrow V_2 = 111 \longrightarrow m = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{111 - 250}{20 - 15} = -27.8$ l_3

$Q_4 \rightarrow t_2 = 25 \rightarrow V_2 = 28 \rightarrow m = \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{28 - 250}{25 - 15} = -22.2$ ℓ_4

$Q_5 \rightarrow t_2 = 30 \rightarrow V_2 = 0 \rightarrow m = \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{0 - 250}{30 - 15} = -16.\dot{6}$ ℓ_5

خمن

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

خمني ميل الخط المستقيم المماس عند النقطه P وذلك بإيجاد متوسط
الميول لاثنين خطين قاطعين

The slope of the tangent line at P =

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$. $y = \frac{1}{1-x}$

(a) If Q is the point $(x, 1/(1 - x))$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 1.5	(ii) 1.9	(iii) 1.99	(iv) 1.999
(v) 2.5	(vi) 2.1	(vii) 2.01	(viii) 2.001

find the slope of the line $PQ \rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$

$P(\underset{x_1}{2}, \underset{y_1}{-1})$ $\qquad Q(\underset{x_2}{x}, \underset{y_2}{\frac{1}{1-x}})$

$$m = \frac{y_2 - y_1}{x_2 x_1} = \frac{\frac{1}{1-x} - (-1)}{x - 2} = \frac{\frac{1}{1-x} + 1 * \frac{1-x}{1-x}}{x-2}$$

$$= \frac{\frac{1}{1-x} + \frac{1(1-x)}{1-x}}{x-2} = \frac{\frac{1+1-x}{1-x}}{x-2}$$

$$= \frac{\frac{2-x}{1-x}}{x-2} = \frac{2-x}{1-x} \div x - 2$$

$$= \frac{2-x}{1-x} * \frac{1}{x-2} = \frac{2-x}{(1-x)(x-2)}$$

$$= \frac{2-x}{x-2-x^2+2x} = \frac{2-x}{-x^2+3x-2}$$

$$m_{tan} = \frac{2-x}{-x^2+3x-2}$$

الصورة العامة للميل
عند النقطة

$P \quad Q$

$(2, -1) \quad (x, \frac{1}{1-x})$

نقطة ثابتة $P(2,-1)$

x	1.5	1.9	1.99	1.999	2.001	2.01	2.1	2.5	متغير
m_{tan}	2	1.111111	1.0101	1.0010	0.999001	0.990	0.90909	0.666667	

$$m_{tan} = \frac{2-x}{-x^2+3x-2}$$

$$x = 1.5 \rightarrow m = \frac{2-1.5}{-(1.5)^2+3(1.5)-2} = 2$$

$$x = 1.9 \rightarrow m = \frac{2-1.9}{-(1.9)^2+3(1.9)-2} = 1.11111$$

$$x = 1.99 \rightarrow m = \frac{2-1.99}{-(1.99)^2+3(1.99)-2} = 1.0101$$

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.

$\boxed{x=2}$

نلاحظ من الفقرة (a) انه كلما اقتربت قيمة x من 2 كلما اقترب الميل من $\boxed{1}$

we guess that slope at point $P(2,-1)$ will be $\boxed{1}$

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$. $m = 1$

$x_1 \quad y_1$

Equation

$y - y_1 = m(x - x_1)$

$y - (-1) = 1(x - 2)$

$y + 1 = x - 2$

$y = x - 2 - 1$

$\boxed{y = x - 3}$

4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.

(a) If Q is the point $(x, \cos \pi x)$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 0	(ii) 0.4	(iii) 0.49
(iv) 0.499	(v) 1	(vi) 0.6
(vii) 0.51	(viii) 0.501	

$\overset{x_1}{}\ \overset{y_1}{}$

$P(0.5, 0)$ $\qquad$ $Q(\overset{x_2}{x}, \overset{y_2}{\cos \pi x})$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\cos \pi x - 0}{x - 0.5} = \frac{\cos \pi x}{x - 0.5}$$

الصيغة
العامة للحل

x	0	0.4	0.49	0.49	0.499	0.501	0.51	0.6	1
m	undefine								

$$x = 0 \longrightarrow m = \frac{\cos \pi x}{x - 0} = \frac{\cos \pi (0)}{0 - 0} \quad \text{undefined}$$

$$x = 0.4 \qquad m = \frac{\cos \pi (0.4)}{0.4 - 0} =$$

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.

guess.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(\underset{x_1}{0.5}, \underset{y_1}{0})$.

$m =$

$y - y_1 = m(x - x_1)$

The Velocity Problem

السرعة اللحظية

Let's consider the *velocity problem:* Find the instantaneous velocity of an object moving along a straight path at a specific time if the position of the object at any time is known. In the next example, we investigate the velocity of a falling ball. Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then (at the earth's surface) Galileo's observation is expressed by the equation

المسافة $\propto$ مربع وقت السقوط

$$\text{المسافة} \quad s(t) = 4.9t^2$$

s

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

حفظ

Find velocity

المسافة بعد الثانية الخامسة

$t_1 = 5 \longrightarrow s(t_1)$

$t_2 = 5.1 \longrightarrow s(t_2)$

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s t_1 - s t_2}{5 - 5.1}$$

الفرق في وقت السقوط

$$= \frac{4.9(5)^2 - 4.9(5.1)^2}{5 - 5.1}$$

$$= 49.49 \text{ m/s}$$

$$s(t) = 4.9t^2$$

distance, بعد الثانية الخامسة

he following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \le t \le 5.1$	49.49
$5 \le t \le 5.05$	49.245
$5 \le t \le 5.01$	49.049
$5 \le t \le 5.001$	49.0049

كثير

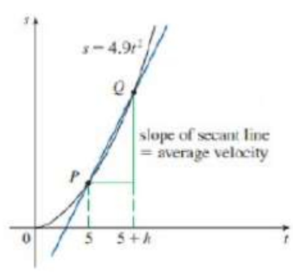
$$\frac{4.9(5)^2 - 4.9(5.05)^2}{5 - 5.05}$$

$$\frac{4.9(5)^2 - 4.9(5.01)^2}{5 - 5.01} =$$

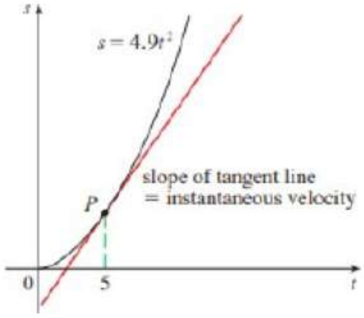
$$\frac{4.9(5)^2 - 4.9(5.001)^2}{5.001}$$

كلما اقتربت من 5 اقتربت

تقترب السرعة من 49



It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the *limiting value* of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus it appears that the (instantaneous) velocity after 5 seconds is 49 m/s. ■



متوسط سرعة الجسم عند
الثانية الخامسة
يقترب من 49m/s

7. The table shows the position of a motorcyclist after accelerating from rest.

الزمن t (seconds)	0	1	2	3	4	5	6
المسافة s (meters)	0	1.5	6.3	14.2	24.1	38.0	53.9

(a) Find the average velocity for each time period:

(i) [2, 4] (ii) [3, 4] (iii) [4, 5] (iv) [4, 6]

الثانية الثانية الرابعة الثانية

$[2,4]$ average velocity $= \dfrac{\text{change in position}}{\text{time elapsed}}$

$= \dfrac{6.3 - 24.1}{2 - 4} = 8.9\ m/s$

$[3,4] \rightarrow$ average velocity $= \dfrac{14.2 - 24.1}{3 - 4} = 9.9\ m/s$

$[4,5] \rightarrow$ average velocity $= \dfrac{24.1 - 38}{4 - 5} = 13.9\ m/s$

$[4,6] \rightarrow$ average velocity $= \dfrac{24.1 - 53.9}{4 - 6} = 14.9\ m/s$

(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

Velocity $[2,3] = \dfrac{6.3 - 14.2}{2 - 3} = 7.9\ m/s$

Velocity $[3,4] = \dfrac{14.2 - 24.1}{3 - 4} = 9.9\ m/s$

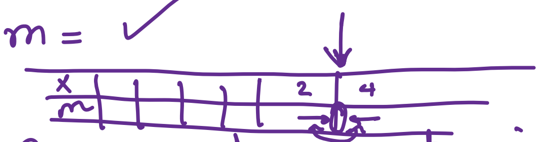
average velocity when $t = 3 \rightarrow \dfrac{7.9 + 9.9}{2} = 8.9\ m/s$

Today's Goal

$\boxed{1}$ $m = \text{slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Given 2 points $P(\underset{x_1}{✓}, \underset{y_1}{✓})$ $Q.(\underset{x_2}{x}, \underset{y_2}{y})$

[illegible] $m =$ ✓



$\boxed{2}$ Guess slope at indicated point

$\boxed{3}$ Equation of tangent line

$y - y_1 = m(x - x_1)$

Point P Given

Velocity

Distance $\propto t^2$

[illegible] $S(t) = 4.9t^2$

average Velocity $= \dfrac{\text{change in Position}}{\text{Time elapsed}}$.